Solutions: Final Exam Review

1 Write down a Riemann Sum with n rectangles for the area above the x-axis and under the curve $y = \sin(x)$ for $0 \leq x \leq \frac{\pi}{2}$. Use right endpoints and Σ notation for your answer.

The interval $0 \leq x \leq \frac{\pi}{2}$ has length $\frac{\pi}{2}$, so each subinterval will have length

$$\Delta x = \frac{\pi}{2n}.$$

The right endpoint of the i^{th} subinterval is

$$x_i = i\Delta x = \frac{\pi i}{2n}.$$

Thus the Riemann sum is

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \sin\left(\frac{\pi i}{2n}\right)\frac{\pi}{2n}.$$

2 Estimate the area under the graph of $y = \ln x$ and above the $x - axis$ for $1 \leq x \leq 2$ using 4 subintervals and **(a)** right endpoints as sample points, and **(b)** left endpoints as sample points. Include 3 decimal places in your answers.

(a) The width of the entire interval is 1, so the width of each subinterval is $\Delta x = \frac{1}{4}$. The right endpoints of the subintervals are $x_1 = \frac{5}{4}$, $x_2 = \frac{3}{2}$, $x_3 = \frac{7}{4}$ and $x_4 = 2$. Therefore, the areas of the approximating rectangles are

$$A_1 = \frac{1}{4}\ln\frac{5}{4}, \quad A_2 = \frac{1}{4}\ln\frac{3}{2}, \quad A_3 = \frac{1}{4}\ln\frac{7}{4}, \quad \text{and} \quad A_4 = \frac{1}{4}\ln 2.$$

Therefore the total area is approximately

$$A_1 + A_2 + A_3 + A_4 = \frac{1}{4}\ln\frac{5}{4} + \frac{1}{4}\ln\frac{3}{2} + \frac{1}{4}\ln\frac{7}{4} + \frac{1}{4}\ln 2 \approx 0.470$$

(b) The width of the entire interval is 1, so the width of each subinterval is $\Delta x = \frac{1}{4}$. The left endpoints of the subintervals are $x_1 = 1$, $x_2 = \frac{5}{4}$, $x_3 = \frac{3}{2}$ and $x_4 = \frac{7}{4}$. Therefore, the areas of the approximating rectangles are

$$A_1 = \frac{1}{4}\ln 1, \quad A_2 = \frac{1}{4}\ln\frac{5}{4}, \quad A_3 = \frac{1}{4}\ln\frac{3}{2}, \quad \text{and} \quad A_4 = \frac{1}{4}\ln\frac{7}{4}.$$

Therefore the total area is approximately

$$A_1 + A_2 + A_3 + A_4 = 0 + \frac{1}{4}\ln\frac{5}{4} + \frac{1}{4}\ln\frac{3}{2} + \frac{1}{4}\ln\frac{7}{4} + \approx 0.297$$

3 Calculate $\int_4^9 \frac{3x-2}{\sqrt{x}}\, dx$. Show all your work.

$$\begin{aligned}
\int_4^9 \frac{3x-2}{\sqrt{x}}\, dx &= \int_4^9 \frac{3x}{\sqrt{x}} - \frac{2}{\sqrt{x}}\, dx \\
&= \int_4^9 3x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}\, dx \\
&= 2x^{\frac{3}{2}} - 4x^{\frac{1}{2}}\Big|_4^9 \\
&= \left(2\cdot 9^{\frac{3}{2}} - 4\cdot 9^{\frac{1}{2}}\right) - \left(2\cdot 4^{\frac{3}{2}} - 4\cdot 4^{\frac{1}{2}}\right) \\
&= (54-12) - (16-8) \\
&= 34.
\end{aligned}$$

4 Calculate $\int x^3 \cos\left(x^4\right)\, dx$. Show all your work.

$$\begin{aligned}
\int x^3 \cos\left(x^4\right)\, dx &= \int \cos u \frac{du}{4} \qquad \left(u = x^4,\ \frac{du}{4} = x^3\, dx\right) \\
&= \frac{1}{4}\int \cos u\, du \\
&= \frac{1}{4}\sin u + C \\
&= \frac{1}{4}\sin x^4 + C.
\end{aligned}$$

5 Calculate $\int \cot\theta\, d\theta$ using substitution. Show all your work.

$$\begin{aligned}
\int \cot\theta\, d\theta &= \int \frac{\cos\theta}{\sin\theta}\, d\theta \\
&= \int \frac{1}{u}\, du \qquad (u = \sin\theta,\ du = \cos\theta\, d\theta) \\
&= \ln|u| + C \\
&= \ln|\sin\theta| + C.
\end{aligned}$$

6 Calculate $\int_1^4 \sqrt{t}\ln t\, dt$. Show all your work.

$$\begin{aligned}
\int \sqrt{t}\ln t\, dt &= \int t^{\frac{1}{2}} \ln t\, dt \\
&= \ln t\left(\frac{2}{3}t^{\frac{3}{2}}\right) - \int \left(\frac{2}{3}t^{\frac{3}{2}}\right)\frac{1}{t}\, dt \qquad \begin{pmatrix} u = \ln t & dv = t^{\frac{1}{2}}\, dt \\ du = \frac{1}{t}\, dt & v = \frac{2}{3}t^{\frac{3}{2}} \end{pmatrix} \\
&= \frac{2}{3}t^{\frac{3}{2}}\ln t - \frac{2}{3}\int t^{\frac{1}{2}}\, dt \\
&= \frac{2}{3}t^{\frac{3}{2}}\ln t - \frac{4}{9}t^{\frac{3}{2}}
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_1^4 \sqrt{t}\ln t\, dt &= \frac{2}{3}t^{\frac{3}{2}}\ln t - \frac{4}{9}t^{\frac{3}{2}}\Big|_1^4 \\
&= \left(\frac{2}{3}\cdot 4^{\frac{3}{2}}\ln 4 - \frac{4}{9}\cdot 4^{\frac{3}{2}}\right) - \left(\frac{2}{3}\cdot 1^{\frac{3}{2}}\ln 1 - \frac{4}{9}\cdot 1^{\frac{3}{2}}\right) \\
&= \frac{16\ln 4}{3} - \frac{28}{9}
\end{aligned}$$

7 Calculate $\int x^2 e^x\, dx$. Show all your work.

$$\begin{aligned}
\int x^2 e^x\, dx &= x^2e^x - \int 2xe^x\, dx \qquad \begin{pmatrix} u = x^2 & dv = e^x\, dx \\ du = 2x\, dx & v = e^x \end{pmatrix} \\
&= x^2e^x - \left(2xe^x - \int 2e^x\, dx\right) \qquad \begin{pmatrix} u = 2x & dv = e^x\, dx \\ du = 2\, dx & v = e^x \end{pmatrix} \\
&= x^2e^x - (2xe^x - 2e^x) + C \\
&= x^2e^x - 2xe^x + 2e^x + C
\end{aligned}$$

8 Use partial fractions to evaluate $\int \frac{1}{x^2+4x+3}\,dx$. Show all your work.

The denominator x^2+4x+3 can be factored as $(x+3)(x+1)$, so we want to decompose the fraction as follows:

$$\frac{1}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}$$

Multiply both sides by the denominator of the left to get

$$1 = A(x+1) + B(x+3).$$

If we plug in $x=-1$ we get the equation

$$1 = 2B,$$

so $B = \frac{1}{2}$. If we plug in $x=-3$ we get the equation

$$1 = -2A,$$

so $A = -\frac{1}{2}$. That is to say,

$$\frac{1}{(x+3)(x+1)} = \frac{\left(-\frac{1}{2}\right)}{x+3} + \frac{\left(\frac{1}{2}\right)}{x+1}.$$

Consequently,

$$\begin{aligned}
\int \frac{1}{x^2+4x+3}\,dx &= \int \frac{1}{(x+3)(x+1)}\,dx \\
&= \int \frac{\left(-\frac{1}{2}\right)}{x+3} + \frac{\left(\frac{1}{2}\right)}{x+1}\,dx \\
&= -\frac{1}{2}\int \frac{1}{x+3}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx \\
&= -\frac{1}{2}\int \frac{1}{s}\,ds + \frac{1}{2}\int \frac{1}{t}\,dt \quad (s = x+3,\ ds = dx \text{ and } t = x+1,\ dt = dx) \\
&= -\frac{1}{2}\ln|s| + \frac{1}{2}\ln|t| + C \\
&= -\frac{1}{2}\ln|x+3| + \frac{1}{2}\ln|x+1| + C.
\end{aligned}$$

9 First use substitution, then integration-by-parts, to calculate the integral $\int \cos\sqrt{x}\,dx$.

$$\begin{aligned}
\int \cos\sqrt{x}\,dx &= \int (\cos s) 2s\,ds \qquad \left(s = \sqrt{x},\ ds = \frac{1}{2\sqrt{x}}\,dx, \text{ or } 2s\,ds = dx\right) \\
&= 2s\sin s - \int 2\sin s\,ds \qquad \begin{pmatrix} u = 2s & dv = \cos s\,ds \\ du = 2\,ds & v = \sin s \end{pmatrix} \\
&= 2s\sin s + 2\cos s + C \\
&= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C.
\end{aligned}$$

10. Find the area bounded between the curves $y = 1 + \sqrt{x}$ and $y = \frac{3+x}{3}$.

First we look for the intersections of these curves:

$$1 + \sqrt{x} = \frac{3+x}{3} \implies 1 + \sqrt{x} = 1 + \frac{x}{3} \implies \sqrt{x} = \frac{x}{3} \implies x = \frac{x^2}{9} \implies x^2 - 9x = 0,$$

so we get $x = 0$ and $x = 9$ as the intersections. On the interval $0 \le x \le 9$, we graph the curves and see that the top curve is $y = 1 + \sqrt{x}$ and the bottom is $y = \frac{3+x}{3}$. Therefore the area betwen them is

$$\begin{aligned}
A &= \int_0^9 (1 + \sqrt{x}) - \left(\frac{3+x}{3}\right)\, dx \\
&= \int_0^9 \sqrt{x} - \frac{x}{3}\, dx \\
&= \frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{6}\Big|_0^9 \\
&= \left(\frac{2}{3}9^{\frac{3}{2}} - \frac{9^2}{6}\right) - \left(\frac{2}{3}0^{\frac{3}{2}} - \frac{0^2}{6}\right) \\
&= \frac{9}{2}.
\end{aligned}$$

11. Find the volume obtained by rotating about the x-axis the region bounded by the curves $y = \frac{1}{4}x^2$ and $y = 5 - x^2$.

Taking vertical cross sections for each fixed value of x, we get "washer" shapes with outer radius $5 - x^2$ and inner radius $\frac{1}{4}x^2$. Thus the area of the crosss sections are $\pi\left((5 - x^2)^2) - (\frac{1}{4}x^2)^2\right)$. The intersections of the curves $y = \frac{1}{4}x^2$ and $y = 5 - x^2$ occur when

$$\frac{1}{4}x^2 = 5 - x^2 \implies \frac{5}{4}x^2 = 5 \implies x^2 = 4 \implies x = \pm 2.$$

Thus the volume is

$$\begin{aligned}
V &= \int_{-2}^{2} \pi\left((5 - x^2)^2) - (\frac{1}{4}x^2)^2\right)\, dx \\
&= \pi \int_{-2}^{2} \left(25 - 10x^2 + \frac{15}{16}x^4\right)\, dx \\
&= \pi\left(25x - \frac{10x^3}{3} + \frac{3x^5}{16}\right)\Big|_{-2}^{2} \\
&= \pi\left(25\,(2) - \frac{10}{3}\,(2)^3 + \frac{3}{16}\,(2)^5\right) \\
&\qquad - \pi\left(25\,(-2) - \frac{10}{3}\,(-2)^3 + \frac{3}{16}\,(-2)^5\right) \\
&= \frac{176\pi}{3}.
\end{aligned}$$

12. Find the volume obtained by rotating about the y-axis the region bounded by the curves $y = x - x^2$ and $y = 0$.

We will use the method of "cylindrical shells". The curves intersect at $x = 0$ and $x = 1$. On the interval $0 \leq x \leq 1$ the top function is $y = x - x^2$ and the bottom function is $y = 0$. Thus the height of a shell is $x - x^2$. The radius of a shell around the y-axis is x. Therefore the volume is

$$\begin{aligned} V &= \int_0^1 2\pi x \left(x - x^2\right) \, dx \\ &= 2\pi \int_0^1 \left(x^2 - x^3\right) \, dx \\ &= 2\pi \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 \\ &= 2\pi \left(\frac{1^3}{3} - \frac{1^4}{4}\right) - 0 \\ &= 2\pi \left(\frac{1}{3} - \frac{1}{4}\right) \\ &= \frac{\pi}{6}. \end{aligned}$$

13. Find the average value of the function $f(x) = x^5 e^{(x^3)}$ on the interval $[0, 2]$.

We're going to need an antiderivative of f, so we find that first.

$$\begin{aligned} \int x^5 e^{(x^3)} \, dx &= \frac{1}{3} \int s e^s \, ds \qquad (s = x^3, \; ds = 3x^2 \, dx) \\ &= \frac{1}{3} \left(s e^s - e^s\right) + C \\ &= \frac{1}{3} \left(x^3 e^{(x^3)} - e^{(x^3)}\right) + C, \end{aligned}$$

where we used integration-by-parts in the second-to-last line. Now we can compute the average value:

$$\begin{aligned} AVERAGE &= \frac{1}{2-0} \int_0^2 x^5 e^{(x^3)} \, dx \\ &= \frac{1}{6} \left(x^3 e^{(x^3)} - e^{(x^3)}\right)\Big|_0^2 \\ &= \frac{1}{6} \left(2^3 e^{(2^3)} - e^{(2^3)}\right) - \frac{1}{6} \left(0^3 e^{(0^3)} - e^{(0^3)}\right) \\ &= \frac{7}{6} e^8 + \frac{1}{6}. \end{aligned}$$

14. The velocity of an object at time t is $v(t) = \frac{1}{t^2-3t}$, and the position at time $t=1$ is $p(1)=1$. Find the position at time $t=2$.

The position is an antiderivative of velocity:

$$\begin{aligned} p(t) &= \int \frac{1}{t^2-3t}\,dt \\ &= \int \frac{1}{t(t-3)}\,dt \\ &= \int \left(\frac{\left(\frac{1}{3}\right)}{t-3} - \frac{\left(\frac{1}{3}\right)}{t}\right)\,dt \qquad \text{(using partial fractions - see below)} \\ &= \frac{1}{3}\ln|t-3| - \frac{1}{3}\ln|t| + C. \end{aligned}$$

We used partial fractions above as follows:

$$\frac{1}{t(t-3)} = \frac{A}{t} + \frac{B}{t-3} \implies 1 = A(t-3) + Bt \implies A = -\frac{1}{3} \text{ and } B = \frac{1}{3}.$$

So we now have $p(t) = \frac{1}{3}\ln|t-3| - \frac{1}{3}\ln|t| + C$. We need to determine what C is. For this, we use the given data: $p(1) = 1$:

$$1 = p(1) = \frac{1}{3}\ln|1-3| - \frac{1}{3}\ln|1| + C = \frac{1}{3}\ln 2 + C \implies C = 1 - \frac{1}{3}\ln 2.$$

Therefore

$$p(t) = \frac{1}{3}\ln|t-3| - \frac{1}{3}\ln|t| + 1 - \frac{1}{3}\ln 2.$$

Consequently,

$$p(2) = \frac{1}{3}\ln|2-3| - \frac{1}{3}\ln|2| + 1 - \frac{1}{3}\ln 2 = \boxed{1 - \tfrac{2}{3}\ln 2}.$$

15. Calculate $\int_1^\infty \frac{\ln x}{x^3}\,dx$.

$$\begin{aligned} \int_1^\infty \frac{\ln x}{x^3}\,dx &= \lim_{T\to\infty} \int_1^T \frac{\ln x}{x^3}\,dx \\ &= \lim_{T\to\infty} \left(-\frac{\ln x}{2x^2} + \int \frac{1}{2x^3}\,dx\right)\Bigg|_1^T \qquad \text{(integration-by-parts)} \\ &= \lim_{T\to\infty} \left(-\frac{\ln x}{2x^2} - \frac{1}{4x^2}\right)\Bigg|_1^T \\ &= \lim_{T\to\infty} \left(-\frac{\ln T}{2T^2} - \frac{1}{4T^2}\right) - \left(-\frac{\ln 1}{2(1)^2} - \frac{1}{4(1)^2}\right) \\ &= \lim_{T\to\infty} -\frac{\ln T}{2T^2} - 0 + 0 + \frac{1}{4} \end{aligned}$$

The remaining limit is an indeterminant form $\left(\frac{\infty}{\infty}\right)$, so we use L'Hospital's rule:

$$\lim_{T\to\infty} \frac{\ln T}{2T^2} = \lim_{T\to\infty} \frac{\left(\frac{1}{T}\right)}{4T} = \lim_{T\to\infty} \frac{1}{4T^2} = 0.$$

Therefore

$$\int_1^\infty \frac{\ln x}{x^3}\,dx = \frac{1}{4}.$$

16. Find the function $y(x)$ which staisfies $\frac{dy}{dx} = \frac{x(y^2+1)}{\sqrt{x^2-1}}$ such that $y = 1$ when $x = \sqrt{2}$.

We use separation of variables:

$$\int \frac{dy}{y^2+1} = \int \frac{x}{\sqrt{x^2-1}}\, dx,$$

so

$$\begin{aligned} \tan^{-1} y &= \frac{1}{2}\int \frac{ds}{\sqrt{s}} \qquad (s = x^2 - 1,\ ds = 2x\ dx) \\ &= \sqrt{s} + C \\ &= \sqrt{x^2-1} + C, \end{aligned}$$

thus

$$y = \tan(\sqrt{x^2-1} + C).$$

The initial condition $y(\sqrt{2}) = 1$ gives us

$$1 = \tan(\sqrt{2-1} + C) = \tan(1 + C),$$

so $C = \tan^{-1}(1) - 1 = \frac{\pi}{4} - 1$. Consequently,

$$y = \tan\left(\sqrt{x^2-1} + \frac{\pi}{4} - 1\right).$$

17. Evaluate the integral $\int \frac{dx}{x^3+x^2}$.

$$\frac{1}{x^3+x^2} = \frac{1}{x^2}x + 1 = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x+1},$$

so

$$1 = A(x+1) + Bx(x+1) + Cx^2.$$

Plugging in $x = 0$ gives $A = 1$. Plugging in $x = -1$ gives $C = 1$. Plugging in $x = 2$ gives

$$1 = A(3) + B(2)(3) + C(4) = 3 + 6B + 4 = 7 + 6B,$$

so $B = -1$. Therefore

$$\frac{1}{x^3+x^2} = \frac{1}{x^2} - \frac{1}{x} + \frac{1}{x+1}.$$

Now we can calculate

$$\begin{aligned} \int \frac{dx}{x^3+x^2} &= \int \frac{1}{x^2} - \frac{1}{x} + \frac{1}{x+1}\, dx \\ &= -\frac{1}{x} - \ln|x| + \ln|x+1| + C. \end{aligned}$$

18. Calculate $\int_0^\pi |\cos(x)|dx$.

Observe that $\cos(x)$ is positive on $0 \le x < \frac{\pi}{2}$ and negative on $\frac{\pi}{2} < x \le \pi$. Therefore

$$\begin{aligned}\int_0^\pi |\cos(x)|dx &= \int_0^{\frac{\pi}{2}} |\cos(x)|dx + \int_{\frac{\pi}{2}}^\pi |\cos(x)|dx \\ &= \int_0^{\frac{\pi}{2}} \cos(x)dx - \int_{\frac{\pi}{2}}^\pi \cos(x)dx \\ &= \sin(x)|_0^{\frac{\pi}{2}} - \sin(x)|_{\frac{\pi}{2}}^{\pi} \\ &= \sin\left(\frac{\pi}{2}\right) - \sin(0) - \sin(\pi) + \sin\left(\frac{\pi}{2}\right) \\ &= 1 - 0 - 0 + 1 \\ &= 2.\end{aligned}$$

19. Solve the initial-value problem $y' = 3y + 1$, $y(0) = -2$.

We separate variables:

$$\int \frac{dy}{3y+1} = \int dx,$$

so

$$\frac{1}{3}\ln|3y+1| = x + C_1.$$

Thus

$$\ln|3y+1| = 3x + C_2.$$

We can use the initial condition right now to solve for C_2:

$$\ln|3(-2)+1| = 3(0) + C_2 \implies \ln 5 = C_2.$$

So we have

$$\ln|3y+1| = 3x + \ln 5,$$

hence

$$|3y+1| = e^{3x+\ln 5} = 5e^{3x}.$$

Therefore

$$3y + 1 = \pm 5e^{3x},$$

so

$$y = -\frac{1}{3} \pm \frac{5}{3}e^{3x}.$$

Because y is a function, there should be only one output for every input, so we need to decide whether to use plus or minus. The initial condition $y(0) = -2$ is less than $-\frac{1}{3}$, so that tells us we must use the solution with the minus sign. Thus we have

$$y = -\frac{1}{3} - \frac{5}{3}e^{3x}.$$

20. Calculate the length of the curve given by the parametric equations

$$x(t) = e^t \sin(t), \qquad y(t) = e^t \cos t, \qquad -\infty < t \leq 0.$$

We have $x'(t) = e^t \sin t + e^t \cos t$ and $y'(t) = e^t \cos t - e^t \sin t$. Therefore we obtain

$$\begin{aligned}
(x'(t))^2 + (y'(t))^2 &= (e^t \sin t + e^t \cos t)^2 + (e^t \cos t - e^t \sin t)^2 \\
&= e^{2t} \sin^2 t + 2e^{2t} \sin t \cos t + e^{2t} \cos^2 t + e^{2t} \cos^2 t - 2e^{2t} \sin t \cos t + e^{2t} \sin^2 t \\
&= 2e^{2t} \sin^2 t + 2e^{2t} \cos^2 t \\
&= 2e^{2t},
\end{aligned}$$

where we have used the pythagorean identity $\sin^2 t + \cos^2 t = 1$. Consequently, the length of the curve is

$$\begin{aligned}
L &= \int_{-\infty}^{0} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\
&= \int_{-\infty}^{0} \sqrt{2e^{2t}}\, dt \\
&= \int_{-\infty}^{0} \sqrt{2}e^t\, dt \\
&= \lim_{T \to -\infty} \int_{T}^{0} \sqrt{2}e^t\, dt \\
&= \lim_{T \to -\infty} \sqrt{2}e^t \Big|_T^0 \\
&= \lim_{T \to -\infty} \sqrt{2}(1 - e^T) \\
&= \sqrt{2}.
\end{aligned}$$