The concept of a function is a unifying theme in the study of mathematics and it has a rich and storied history. The word “function” was first coined by Gottfried Wilhelm Leibniz (c. 1694) (one of the co-founders of calculus with Sir Isaac Newton). Leibniz’s concept of function was relegated to how geometrical properties of a curve (e.g., subtangents and subnormals) depended on the shape of the curve. Johann Bernoulli (c. 1718) described a function of a variable as a quantity that is constructed from that variable and some constants. Indeed, even Leonard Euler (1707-1793), who was a former student of Bernoulli, described the dependence of one variable on another through the means of an analytical expression. In his *Introductio in Analysin Infinitorum* (Introduction to infinite analyses) (1748), Euler states:

*The nature of the curve, provided it is continuous, is expressed through the quality of the function $y$, that is, the rule of formation whereby the value of $y$ is obtained from the composition of constants and the variable $x$."

Euler equated the word function with an analytic equation describing the relationship between the independent and dependent variables. This is not the modern definition of a function, but it is precisely how many of today’s students think about the concept of a function; i.e., a function is an equation.

Euler’s definition of function did not change much until mathematicians began studying the equation of the vibrating string, an equation known as the wave equation. Jean Baptiste Fourier (1768-1830), in his classic work on heat transfer, claimed that any function could be expressed as an infinite series of trigonometric functions. It turned out that he was wrong, and it was up to Johann Peter Gustav Lejeune Dirichlet (1805-1859) to set sufficient conditions on functions to correct Fourier’s error. In order to do that, Dirichlet had to separate the concept of function from its dependence on an analytic expression. Dirichlet’s definition of a function closely mirrors the modern day definition.

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2.1 Introduction to Functions

Our development of the function concept is a modern one, but quite quick, particularly in light of the fact that today’s definition took over 300 years to reach its present state. We begin with the definition of a relation.

**Relations**

We use the notation \((2, 4)\) to denote what is called an *ordered pair*. If you think of the positions taken by the ordered pairs \((4, 2)\) and \((2, 4)\) in the coordinate plane (see Figure 1), then it is immediately apparent why order is important. The ordered pair \((4, 2)\) is simply not the same as the ordered pair \((2, 4)\).

The first element of an ordered pair is called its *abscissa*. The second element of an ordered pair is called its *ordinate*. Thus, for example, the abscissa of \((4, 2)\) is 4, while the ordinate of \((4, 2)\) is 2.

**Definition 1.** A collection of ordered pairs is called a *relation*.

For example, the collection of ordered pairs

\[ R = \{(0, 1), (0, 2), (3, 4)\} \]  

is a relation.

**Definition 3.** The **domain** of a relation is the collection of all abscissas of each ordered pair.

Thus, the domain of the relation \(R\) in (2) is

\[ \text{Domain} = \{0, 3\} \]

Note that we list each abscissa only once.

**Definition 4.** The **range** of a relation is the collection of all ordinates of each ordered pair.

Thus, the range of the relation \(R\) in (2) is

\[ \text{Range} = \{1, 2, 4\} \]

Let’s look at an example.

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Example 5. Consider the relation $T$ defined by
\[ T = \{(1, 2), (3, 2), (4, 5)\}. \] (6)

What are the domain and range of this relation?

The domain is the collection of abscissas of each ordered pair. Hence, the domain of $T$ is

\[ \text{Domain} = \{1, 3, 4\}. \]

The range is the collection of ordinates of each ordered pair. Hence, the range of $T$ is

\[ \text{Range} = \{2, 5\}. \]

Note that we list each ordinate in the range only once.

In Example 5, the relation is described by listing the ordered pairs. This is not the only way that one can describe a relation. For example, a graph certainly represents a collection of ordered pairs.

Example 7. Consider the graph of the relation $S$ shown in Figure 2.

What are the domain and range of the relation $S$?

There are five ordered pairs (points) plotted in Figure 2. They are

\[ S = \{(1, 2), (2, 1), (2, 4), (3, 3), (4, 4)\}. \]

Therefore, the relation $S$ has Domain $= \{1, 2, 3, 4\}$ and Range $= \{1, 2, 3, 4\}$. In the case of the range, note how we’ve sorted the ordinates of each ordered pair in ascending order, taking care not to list any ordinate more than once.
Functions

A function is a very special type of relation. We begin with a formal definition.

**Definition 8.** A relation is a function if and only if each object in its domain is paired with one and only one object in its range.

This is not an easy definition, so let’s take our time and consider a few examples. Consider, if you will, the relation $R$ in (2), repeated here again for convenience.

$$R = \{(0,1), (0,2), (3,4)\}$$

The domain is $\{0,3\}$ and the range is $\{1,2,4\}$. Note that the number 0 in the domain of $R$ is paired with two numbers from the range, namely, 1 and 2. Therefore, $R$ is not a function.

There is a construct, called a mapping diagram, which can be helpful in determining whether a relation is a function. To craft a mapping diagram, first list the domain on the left, then the range on the right, then use arrows to indicate the ordered pairs in your relation, as shown in **Figure 3**.

![Figure 3](image)

It’s clear from the mapping diagram in **Figure 3** that the number 0 in the domain is being paired (mapped) with two different range objects, namely, 1 and 2. Thus, $R$ is not a function.

Let’s look at another example.

**Example 9.** Is the relation described in **Example 5** a function?

First, let’s repeat the listing of the relation $T$ here for convenience.

$$T = \{(1,2), (3,2), (4,5)\}$$

Next, construct a mapping diagram for the relation $T$. List the domain on the left, the range on the right, then use arrows to indicate the pairings, as shown in **Figure 4**.

From the mapping diagram in **Figure 4**, we can see that each domain object on the left is paired (mapped) with exactly one range object on the right. Hence, the relation $T$ is a function.
Let’s look at another example.

**Example 10.** *Is the relation of Example 7, pictured in Figure 2, a function?*

First, we repeat the graph of the relation from Example 7 here for convenience. This is shown in Figure 5(a). Next, we list the ordered pairs of the relation $S$.

$$S = \{(1, 2), (2, 1), (2, 4), (3, 3), (4, 4)\}$$

Then we create a mapping diagram by first listing the domain on the left, the range on the right, then using arrows to indicate the pairings, as shown in Figure 5(b).

Each object in the domain of $S$ gets mapped to exactly one range object with one exception. The domain object 2 is paired with two range objects, namely, 1 and 4. Consequently, $S$ is **not** a function.

This is a good point to summarize what we’ve learned about functions thus far.
Summary 11. A function consists of three parts:
1. a set of objects which mathematicians call the domain,
2. a second set of objects which mathematicians call the range,
3. and a rule that describes how to assign a unique range object to each object in the domain.

The rule can take many forms. For example, we can use sets of ordered pairs, graphs, and mapping diagrams to describe the function. In the sections that follow, we will explore other ways of describing a function, for example, through the use of equations and simple word descriptions.

Function Notation

We’ve used the word “mapping” several times in the previous examples. This is not a word to be taken lightly; it is an important concept. In the case of the mapping diagram in Figure 5(b), we would say that the number 1 in the domain of $S$ is “mapped” (or “sent”) to the number 2 in the range of $S$.

There are a number of different notations we could use to indicate that the number 1 in the domain is “mapped” or “sent” to the number 2 in the range. One possible notation is

$$S : 1 \rightarrow 2,$$

which we would read as follows: “The relation $S$ maps (sends) 1 to 2.” In a similar vein, we see in Figure 5(b) that the domain objects 3 and 4 are mapped (sent) to the range objects 3 and 4, respectively. In symbols, we would write

$$S : 3 \rightarrow 3,$$

$$S : 4 \rightarrow 4.$$

A difficulty arises when we examine what happens to the domain object 2. There are two possibilities, either

$$S : 2 \rightarrow 1,$$

or

$$S : 2 \rightarrow 4.$$

Which should we choose? The 1? Or the 4? Thus, $S$ is not well-defined and is not a function, since we don’t know which range object to pair with the domain object 1.

The idea of mapping gives us an alternative way to describe a function. We could say that a function is a rule that assigns a unique object in its range to each object in its domain. Take for example, the function that maps each real number to its square. If we name the function $f$, then $f$ maps 5 to 25, 6 to 36, $-7$ to 49, and so on. In symbols, we would write
Chapter 2 Functions

In general, we could write

\[ f : x \rightarrow x^2. \]

Note that each real number \( x \) gets mapped to a unique number in the range of \( f \), namely, \( x^2 \). Consequently, the function \( f \) is well defined. We’ve succeeded in writing a rule that completely defines the function \( f \).

As another example, let’s define a function that takes a real number, doubles it, then adds 3. If we name the function \( g \), then \( g \) would take the number 7, double it, then add 3. That is,

\[ g : 7 \rightarrow 2(7) + 3 \]

Simplifying, \( g : 7 \rightarrow 17 \). Similarly, \( g \) would take the number 9, double it, then add 3. That is,

\[ g : 9 \rightarrow 2(9) + 3 \]

Simplifying, \( g : 9 \rightarrow 21 \). In general, \( g \) takes a real number \( x \), doubles it, then adds three. In symbols, we would write

\[ g : x \rightarrow 2x + 3. \]

Notice that each real number \( x \) is mapped by \( g \) to a unique number in its range. Therefore, we’ve again defined a rule that completely defines the function \( g \).

It is helpful to think of a function as a machine. The machine receives input, processes it according to some rule, then outputs a result. Something goes in (input), then something comes out (output). In the case of the function described by the rule \( f : x \rightarrow x^2 \), the “\( f \)-machine” receives input \( x \), then applies its “square rule” to the input and outputs \( x^2 \), as shown in Figure 6(a). In the case of the function described by the rule \( g : x \rightarrow 2x + 3 \), the “\( g \)-machine” receives input \( x \), then applies the rules “double,” then “add 3,” in that order, then outputs \( 2x + 3 \), as shown in Figure 6(b).

\[ \]

Figure 6. Function machines.
Let’s look at another example.

**Example 12.** Suppose that $f$ is defined by the following rule. For each real number $x$,

$$f : x \rightarrow x^2 - 2x - 3.$$

Where does $f$ map the number $-3$? Is $f$ a function?

We substitute $-3$ for $x$ in the rule for $f$ and obtain

$$f : -3 \rightarrow (-3)^2 - 2(-3) - 3.$$  

Simplifying,

$$f : -3 \rightarrow 9 + 6 - 3,$$

or,

$$f : -3 \rightarrow 12.$$

Thus, $f$ maps (sends) the number $-3$ to the number 12. It should be clear that each real number $x$ will be mapped (sent) to a unique real number, as defined by the rule $f : x \rightarrow x^2 - 2x - 3$. Therefore, $f$ is a function.

Let’s look at another example.

**Example 13.** Suppose that $g$ is defined by the following rule. For each real number $x$ that is greater than or equal to zero,

$$g : x \rightarrow \pm \sqrt{x}.$$

Where does $g$ map the number 4? Is $g$ a function?

Again, we substitute 4 for $x$ in the rule for $g$ and obtain

$$g : 4 \rightarrow \pm \sqrt{4}.$$  

Simplifying,

$$g : 4 \rightarrow \pm 2.$$  

Thus, $g$ maps (sends) the number 4 to **two** different objects in its range, namely, 2 and $-2$. Consequently, $g$ is not well-defined and is **not** a function.

Let’s look at another example.

**Example 14.** Suppose that we have functions $f$ and $g$, defined by

$$f : x \rightarrow x^4 + 11 \quad \text{and} \quad g : x \rightarrow (x + 2)^2.$$
Where does $g$ send 5?

In this example, we see a clear advantage of function notation. Because our functions have distinct names, we can simply reference the name of the function we want our readers to use. In this case, we are asked where the function $g$ sends the number 5, so we substitute 5 for $x$ in

$$g : x \rightarrow (x + 2)^2.$$  

That is,

$$g : 5 \rightarrow (5 + 2)^2.$$  

Simplifying, $g : 5 \rightarrow 49$.

Modern Notation

Function notation is relatively new, with some of the earliest symbolism first occurring in the 17th century. In a letter to Leibniz (1698), Johann Bernoulli wrote “For denoting any function of a variable quantity $x$, I rather prefer to use the capital letter having the same name $X$ or the Greek $\xi$, for it appears at once of what variable it is a function; this relieves the memory.”

Mathematicians are fond of the notation

$$f : x \rightarrow x^2 - 2x,$$

because it conveys a sense of what a function does; namely, it “maps” or “sends” the number $x$ to the number $x^2 - 2x$. This is what functions do, they pair each object in their domain with a unique object in their range. Equivalently, functions “send” each object in their domain to a unique object in their range.

However, in common computational situations, the “arrow” notation can be a bit clumsy, so mathematicians tend to favor a slightly different notation. Instead of writing

$$f : x \rightarrow x^2 - 2x,$$

mathematicians tend to favor the notation

$$f(x) = x^2 - 2x.$$  

It is important to understand from the outset that these two different notations are equivalent; they represent the same function $f$, one that pairs each real number $x$ in its domain with the real number $x^2 - 2x$ in its range.

The first notation, $f : x \rightarrow x^2 - 2x$, conveys the sense that the function $f$ is a mapping. If we read this notation aloud, we should pronounce it as “$f$ sends (or maps) $x$ to $x^2 - 2x$.” The second notation, $f(x) = x^2 - 2x$, is pronounced “$f$ of $x$ equals $x^2 - 2x$.”
Warning 15. The phrase “$f$ of $x$” is unfortunate, as our readers might recall being trained from a very early age to pair the word “of” with the operation of multiplication. For example, $1/2$ of $12$ is $6$, as in $1/2 \times 12 = 6$. However, in the context of function notation, even though $f(x)$ is read aloud as “$f$ of $x$,” it does not mean “$f$ times $x$.” Indeed, if we remind ourselves that the notation $f(x) = x^2 - 2x$ is equivalent to the notation $f : x \rightarrow x^2 - 2x$, then even though we might say “$f$ of $x$,” we should be thinking “$f$ sends $x$” or “$f$ maps $x$.“ We should not be thinking “$f$ times $x$.”

Now, let’s see how each of these notations operates on the number $5$. In the first case, using the “arrow” notation,

$$f : x \rightarrow x^2 - 2x.$$ 

To find where $f$ sends $5$, we substitute $5$ for $x$ as follows.

$$f : 5 \rightarrow (5)^2 - 2(5).$$

Simplifying, $f : 5 \rightarrow 15$. Now, because both notations are equivalent, to compute $f(5)$, we again substitute $5$ for $x$ in

$$f(x) = x^2 - 2x.$$ 

Thus,

$$f(5) = (5)^2 - 2(5).$$

Simplifying, $f(5) = 15$. This result is read aloud as “$f$ of $5$ equals $15$,” but we want to be thinking “$f$ sends $5$ to $15$.”

Let’s look at examples that use this modern notation.

**Example 16.** Given $f(x) = x^3 + 3x^2 - 5$, determine $f(-2)$.

Simply substitute $-2$ for $x$. That is,

$$f(-2) = (-2)^3 + 3(-2)^2 - 5$$

$$= -8 + 3(4) - 5$$

$$= -8 + 12 - 5$$

$$= -1.$$ 

Thus, $f(-2) = -1$. Again, even though this is pronounced “$f$ of $-2$ equals $-1$,” we still should be thinking “$f$ sends $-2$ to $-1$.”
Example 17. Given

\[ f(x) = \frac{x + 3}{2x - 5}, \]

determine \( f(6) \).

Simply substitute 6 for \( x \). That is,

\[ f(6) = \frac{6 + 3}{2(6) - 5} = \frac{9}{12 - 5} = \frac{9}{7}. \]

Thus, \( f(6) = \frac{9}{7} \). Again, even though this is pronounced “\( f \) of 6 equals 9/7,” we should still be thinking “\( f \) sends 6 to 9/7.”

Example 18. Given \( f(x) = 5x - 3 \), determine \( f(a + 2) \).

If we’re thinking in terms of mapping notation, then

\[ f : x \rightarrow 5x - 3. \]

Think of this mapping as a “machine.” Whatever we put into the machine, it first is multiplied by 5, then 3 is subtracted from the result, as shown in Figure 7. For example, if we put a 4 into the machine, then the function rule requires that we multiply input 4 by 5, then subtract 3 from the result. That is,

\[ f : 4 \rightarrow 5(4) - 3. \]

Simplifying, \( f : 4 \rightarrow 17 \).

Figure 7. The multiply by 5 then subtract 3 machine.
Similarly, if we put an $a + 2$ into the machine, then the function rule requires that we multiply the input $a + 2$ by 5, then subtract 3 from the result. That is,

$$f : a + 2 \rightarrow 5(a + 2) - 3.$$ 

Using modern function notation, we would write

$$f(a + 2) = 5(a + 2) - 3.$$ 

Note that this is again a simple substitution, where we replace each occurrence of $x$ in the formula $f(x) = 5x - 3$ with the expression $a + 2$. Finally, use the distributive property to first multiply by 5, then subtract 3.

$$f(a + 2) = 5a + 10 - 3 = 5a + 7.$$ 

We will often need to substitute the result of one function evaluation into a second function for evaluation. Let’s look at an example.

**Example 19.** Given two functions defined by $f(x) = 3x + 2$ and $g(x) = 5 - 4x$, find $f(g(2))$.

The nested parentheses in the expression $f(g(2))$ work in the same manner that they do with nested expressions. The rule is to work the innermost grouping symbols first, proceeding outward as you work. We’ll first evaluate $g(2)$, then evaluate $f$ at the result.

We begin. First, evaluate $g(2)$ by substituting 2 for $x$ in the defining equation $g(x) = 5 - 4x$. Note that $g(2) = 5 - 4(2)$, then simplify.

$$f(g(2)) = f(5 - 4(2)) = f(5 - 8) = f(-3)$$

To complete the evaluation, we substitute $-3$ for $x$ in the defining equation $f(x) = 3x + 2$, then simplify.

$$f(-3) = 3(-3) + 2 = -9 + 2 = -7.$$ 

Hence, $f(g(2)) = -7$.

It is conventional to arrange the work in one contiguous block, as follows.

$$f(g(2)) = f(5 - 4(2))$$
$$= f(-3)$$
$$= 3(-3) + 2$$
$$= -7$$

You can shorten the task even further if you are willing to do the function substitution and simplification in your head. First, evaluate $g$ at 2, then $f$ at the result.

$$f(g(2)) = f(-3) = -7$$
Let’s look at another example of this unique way of combining functions.

**Example 20.** Given $f(x) = 5x + 2$ and $g(x) = 3 - 2x$, evaluate $g(f(a))$ and simplify the result.

We work the inner function evaluation in the expression $g(f(a))$ first. Thus, to evaluate $f(a)$, we substitute $a$ for $x$ in the definition $f(x) = 5x + 2$ to get

$$g(f(a)) = g(5a + 2).$$

Now we need to evaluate $g(5a + 2)$. To do this, we substitute $5a + 2$ for $x$ in the definition $g(x) = 3 - 2x$ to get

$$g(5a + 2) = 3 - 2(5a + 2).$$

We can expand this last result and simplify. Thus,

$$g(f(a)) = 3 - 10a - 4 = -10a - 1.$$  

Again, it is conventional to arrange the work in one continuous block, as follows.

$$g(f(a)) = g(5a + 2)$$
$$= 3 - 2(5a + 2)$$
$$= 3 - 10a - 4$$
$$= -10a - 1$$

Hence, $g(f(a)) = -10a - 1$.

**Extracting the Domain of a Function**

We’ve seen that the domain of a relation or function is the set of all the first coordinates of its ordered pairs. However, if a functional relationship is defined by an equation such as $f(x) = 3x - 4$, then it is not practical to list all ordered pairs defined by this relationship. For any real $x$-value, you get an ordered pair. For example, if $x = 5$, then $f(5) = 3(5) - 4 = 11$, leading to the ordered pair $(5, f(5))$ or $(5, 11)$. As you can see, the number of such ordered pairs is infinite. For each new $x$-value, we get another function value and another ordered pair.

Therefore, it is easier to turn our attention to the values of $x$ that yield real number responses in the equation $f(x) = 3x - 4$. This leads to the following key idea.

**Definition 21.** If a function is defined by an equation, then the domain of the function is the set of “permissible $x$-values,” the values that produce a real number response defined by the equation.
We sometimes like to say that the domain of a function is the set of “OK $x$-values to use in the equation.” For example, if we define a function with the rule $f(x) = 3x - 4$, it is immediately apparent that we can use any value we want for $x$ in the rule $f(x) = 3x - 4$. Thus, the domain of $f$ is all real numbers. We can write that the domain $D = \mathbb{R}$, or we can use interval notation and write that the domain $D = (-\infty, \infty)$.

It is not the case that $x$ can be any real number in the function defined by the rule $f(x) = \sqrt{x}$. It is not possible to take the square root of a negative number. Therefore, $x$ must either be zero or a positive real number. In set-builder notation, we can describe the domain with $D = \{x : x \geq 0\}$. In interval notation, we write $D = [0, \infty)$.

We must also be aware of the fact that we cannot divide by zero. If we define a function with the rule $f(x) = x/(x - 3)$, we immediately see that $x = 3$ will put a zero in the denominator. Division by zero is not defined. Therefore, 3 is not in the domain of $f$. No other $x$-value will cause a problem. The domain of $f$ is best described with set-builder notation as $D = \{x : x \neq 3\}$.

**Functions Without Formulae**

In the previous section, we defined functions by means of a formula, for example, as in

$$f(x) = \frac{x + 3}{2 - 3x}.$$  

Euler would be pleased with this definition, for as we have said previously, Euler thought of functions as analytic expressions.

However, it really isn’t necessary to provide an expression or formula to define a function. There are other forms we can use to express a functional relationship: a graph, a table, or even a narrative description. The only thing that is really important is the requirement that the function be well-defined, and by “well-defined,” we mean that each object in the function’s domain is paired with one and only one object in its range.

As an example, let’s look at a special function $\pi$ on the natural numbers, which returns the number of primes less than or equal to a given natural number. For example, the primes less than or equal to the number 23 are 2, 3, 5, 7, 11, 13, 17, 19, and 23, nine numbers in all. Therefore, the number of primes less than or equal to 23 is nine. In symbols, we would write

$$\pi(23) = 9.$$  

---

2. The square of a real number is either zero or a positive real number. It is not possible to square a real number and get a negative result. Therefore, there is no real square root of a negative number.

3. The use of $\pi$ in this context is unfortunate and apt to confuse. Readers are more likely to associate the symbol $\pi$ with the formulae for finding the area and circumference of a circle, with approximate value $\pi \approx 3.14159 \ldots$. As John Derbyshire states in *Prime Obsession*, “The Greek alphabet has only 24 letters and by the time mathematicians got round to giving this function a symbol (the person responsible in this case is Edmund Landau, in 1909), all 24 had been pretty much used up and they had to start recycling them.” In short, the symbol is standard, so we’ll just have to live with it.
Note the absence of a formula in the definition of this function. Indeed, the definition is descriptive in nature, so we might write

$$\pi(n) = \text{number of primes less than or equal to } n.$$ 

The important thing is not how we define this special function $\pi$, but the fact that it is well-defined; that is, for each natural number $n$, there are a fixed number of primes less than or equal to $n$. Thus, each natural number in the domain of $\pi$ is paired with one and only one number in its range.

Now, just because our function doesn’t provide an expression for calculating the number of primes less than or equal to a given natural number $n$, it doesn’t stop mathematicians from seeking such a formula. Euclid of Alexandria (325-265 BC), a Greek mathematician, proved that the number of primes is infinite, but it was the German mathematician and scientist, Johann Carl Friedrich Gauss (1777-1855), who first proposed that the number of primes less than or equal to $n$ can be approximated by the formula

$$\pi(n) \approx \frac{n}{\ln n},$$

where $\ln n$ is the “natural logarithm” of $n$ (to be explained in Chapter 9). This approximation gets better and better with larger and larger values of $n$. The formula was refined by Gauss, who did not provide a proof, and the problem became known as the Prime Number Theorem. It was not until 1896 that Jacques Salomon Hadamard (1865-1963) and Charles Jean Gustave Nicolas Baron de la Vallee Poussin (1866-1962), working independently, provided a proof of the Prime Number Theorem.
2.1 Exercises

In Exercises 1-6, state the domain and range of the given relation.

1. \( R = \{(1,3), (2,4), (3,4)\} \)
2. \( R = \{(1,3), (2,4), (2,5)\} \)
3. \( R = \{(1,4), (2,5), (2,6)\} \)
4. \( R = \{(1,5), (2,4), (3,6)\} \)
5. 

In Exercises 7-12, create a mapping diagram for the given relation and state whether or not it is a function.

7. The relation in Exercise 1.
8. The relation in Exercise 2.
11. The relation in Exercise 5.

13. Given that \( g \) takes a real number and doubles it, then \( g : x \rightarrow ? \).
14. Given that \( f \) takes a real number and divides it by 3, then \( f : x \rightarrow ? \).
15. Given that \( g \) takes a real number and adds 3 to it, then \( g : x \rightarrow ? \).
16. Given that \( h \) takes a real number and subtracts 4 from it, then \( h : x \rightarrow ? \).
17. Given that \( g \) takes a real number, doubles it, then adds 5, then \( g : x \rightarrow ? \).
18. Given that \( h \) takes a real number, subtracts 3 from it, then divides the result by 4, then \( h : x \rightarrow ? \).

Given that the function \( f \) is defined by the rule \( f : x \rightarrow 3x - 5 \), determine where the input number is mapped in Exercises 19-22.

19. \( f : 3 \rightarrow ? \)

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20. \( f : -5 \rightarrow ? \)
21. \( f : a \rightarrow ? \)
22. \( f : 2a + 3 \rightarrow ? \)

Given that the function \( f \) is defined by the rule \( f : x \rightarrow 4 - 5x \), determine where the input number is mapped in Exercises 23-26.

23. \( f : 2 \rightarrow ? \)
24. \( f : -3 \rightarrow ? \)
25. \( f : a \rightarrow ? \)
26. \( f : 2a + 11 \rightarrow ? \)

Given that the function \( f \) is defined by the rule \( f : x \rightarrow x^2 - 4x - 6 \), determine where the input number is mapped in Exercises 27-30.

27. \( f : 1 \rightarrow ? \)
28. \( f : -2 \rightarrow ? \)
29. \( f : -1 \rightarrow ? \)
30. \( f : a \rightarrow ? \)

Given that the function \( f \) is defined by the rule \( f : x \rightarrow 3x - 9 \), determine where the input number is mapped in Exercises 31-34.

31. \( f : a \rightarrow ? \)
32. \( f : a + 1 \rightarrow ? \)
33. \( f : 2a - 5 \rightarrow ? \)
34. \( f : a + h \rightarrow ? \)

Given that the functions \( f \) and \( g \) are defined by the rules \( f : x \rightarrow 2x + 3 \) and \( g : x \rightarrow 4 - x \), determine where the input number is mapped in Exercises 35-38.

35. \( f : 2 \rightarrow ? \)
36. \( g : 2 \rightarrow ? \)
37. \( f : a + 1 \rightarrow ? \)
38. \( g : a - 3 \rightarrow ? \)

39. Given that \( g \) takes a real number and triples it, then \( g(x) = ? \).

40. Given that \( f \) takes a real number and divides it by 5, then \( f(x) = ? \).

41. Given that \( g \) takes a real number and subtracts it from 10, then \( g(x) = ? \).

42. Given that \( f \) takes a real number, multiplies it by 5 and then adds 4 to the result, then \( f(x) = ? \).

43. Given that \( f \) takes a real number, doubles it, then subtracts the result from 11, then \( f(x) = ? \).

44. Given that \( h \) takes a real number, doubles it, adds 5, then takes the square root of the result, then \( h(x) = ? \).

In Exercises 45-54, evaluate the given function at the given value \( b \).

45. \( f(x) = 12x + 2 \) for \( b = 6 \).
46. \( f(x) = -11x - 4 \) for \( b = -3 \).
47. \( f(x) = -9x - 1 \) for \( b = -5 \).
48. \( f(x) = 11x + 4 \) for \( b = -4 \).
49. \( f(x) = 4 \) for \( b = -12 \).
50. \( f(x) = 7 \) for \( b = -7 \).
51. \( f(x) = 0 \) for \( b = -7 \).
52. \( f(x) = 12x + 8 \) for \( b = -3 \).
53. \( f(x) = -9x + 3 \) for \( b = -1 \).
54. \( f(x) = 6x - 3 \) for \( b = 3 \).

In Exercises 55-58, given that the function \( f \) is defined by the rule \( f(x) = 2x + 7 \) determine where the input number is mapped.

55. \( f(a) = \) ?
56. \( f(a + 1) = \) ?
57. \( f(3a - 2) = \) ?
58. \( f(a + h) = \) ?

In Exercises 59-62, given that the function \( g \) is defined by the rule \( g(x) = 3 - 2x \) determine where the input number is mapped.

59. \( g(a) = \) ?
60. \( g(a + 3) = \) ?
61. \( g(2 - 5a) = \) ?
62. \( g(a + h) = \) ?

Given that the functions \( f \) and \( g \) are defined by the rules \( f(x) = 3x + 4 \) and \( g(x) = 2x - 5 \), determine where the input number is mapped in Exercises 67-70.

67. \( f(g(2)) = \) ?
68. \( g(f(2)) = \) ?
69. \( f(g(a)) = \) ?
70. \( g(f(a)) = \) ?

Given that the functions \( f \) and \( g \) are defined by the rules \( f(x) = 2x - 9 \) and \( g(x) = 11 \), determine where the input number is mapped in Exercises 71-74.

71. \( f(g(2)) = \) ?
72. \( g(f(2)) = \) ?
73. \( f(g(a)) = \) ?
74. \( g(f(a)) = \) ?

Use set-builder notation to describe the domain of each of the functions defined in Exercises 75-78.

75. \( f(x) = \frac{93}{x + 98} \)
76. \( f(x) = \frac{54}{x + 65} \)
77. \( f(x) = \frac{87}{x - 88} \)
78. \( f(x) = \frac{30}{x - 52} \)
Use set-builder and interval notation to describe the domain of the functions defined in Exercises 79-82.

79. \( f(x) = \sqrt{x + 69} \)

80. \( f(x) = \sqrt{x + 62} \)

81. \( f(x) = \sqrt{x - 81} \)

82. \( f(x) = \sqrt{x - 98} \)

Two integers are said to be \textit{relatively prime} if their greatest common divisor is 1. For example, the greatest common divisor of 6 and 35 is 1, so 6 and 35 are relatively prime. On the other hand, the greatest common divisor of 14 and 21 is \textbf{not} 1 (it is 7), so 14 and 21 are \textbf{not} relatively prime. The \textit{Euler \( \phi \)-function} is defined as follows:

- If \( n = 1 \), then \( \phi(n) = 1 \).
- If \( n > 1 \), then \( \phi(n) \) is the number of positive integers less than \( n \) that are relatively prime to \( n \). In Exercises 83-84, evaluate the Euler \( \phi \)-function at the given input.

83. \( \phi(12) \)

84. \( \phi(36) \)

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2.1 Answers

1. Domain = \{1, 2, 3\}, Range = \{3, 4\}

3. Domain = \{1, 2\}, Range = \{4, 5, 6\}

5. Domain = \{1, 2, 3\}, Range = \{1, 2, 3, 4\}

7. Function.

9.

11.

13. \( g : x \rightarrow 2x \)

15. \( g : x \rightarrow x + 3 \)

17. \( g : x \rightarrow 2x + 5 \)

19. \( f : 3 \rightarrow 4 \)

21. \( f : a \rightarrow 3a - 5 \)

23. \( f : 2 \rightarrow -6 \)

25. \( f : a \rightarrow 4 - 5a \)

27. \( f : 1 \rightarrow -9 \)

29. \( f : -1 \rightarrow -1 \)

31. \( f : a \rightarrow 3a - 9 \)

33. \( f : 2a - 5 \rightarrow 6a - 24 \)

35. \( f : 2 \rightarrow 7 \)

37. \( f : a + 1 \rightarrow 2a + 5 \)

39. \( g(x) = 3x \)

41. \( g(x) = 10 - x \)

43. \( f(x) = 11 - 2x \)

45. 74

47. 44

49. 4

51. 0

53. 12

55. \( f(a) = 2a + 7 \)

57. \( f(3a - 2) = 6a + 3 \)

59. \( g(a) = 3 - 2a \)

61. \( g(2 - 5a) = 10a - 1 \)

63. \( f(a) = 1 - a \)

65. \( f(a + 3) = -a - 2 \)
67. $f(g(2)) = 1$
69. $f(g(a)) = 6a - 11$
71. $f(g(2)) = 13$
73. $f(g(a)) = 13$
75. Domain = \{x : x \neq -98\}$
77. Domain = \{x : x \neq 88\}
79. Domain = $[-69, \infty) = \{x : x \geq -69\}$
81. Domain = $[81, \infty) = \{x : x \geq 81\}$
83. $\phi(12) = 4$
### 2.2 The Graph of a Function

Rene Descartes (1596-1650) was a French philosopher and mathematician who is well known for the famous phrase “cogito ergo sum” (I think, therefore I am), which appears in his *Discours de la methode pour bien conduire sa raison, et chercher la verite dans les sciences* (Discourse on the Method of Rightly Conducting the Reason, and Seeking Truth in the Sciences). In that same treatise, Descartes introduces his coordinate system, a method for representing points in the plane via pairs of real numbers. Indeed, the Cartesian plane of modern day is so named in honor of Rene Descartes, who some call the “Father of Modern Mathematics.”

Descartes’ work, which forever linked geometry and algebra, was continued in an appendix to *Discourse on Method*, entitled *La Geometrie*, which some consider the beginning of modern mathematics. Certainly both Newton and Leibniz, in developing the Calculus, built upon the foundation provided in this work by Descartes.

A Cartesian Coordinate System consists of a pair of axes, usually drawn at right angles to one another in the plane, one horizontal (labeled $x$) and one vertical (labeled $y$), as shown in the Figure 1. The quadrants are numbered I, II, III, and IV, in counterclockwise order, and samples of ordered pairs of the form $(x, y)$ are shown in each quadrant of the Cartesian coordinate system in Figure 1.

![Figure 1. The Cartesian coordinate system.](http://msenux.redwoods.edu/IntAlgText/)

Now, suppose that we have a relation

\[ R = \{(1, 2), (3, 1), (3, 4), (4, 3)\}. \]

Recall that *relation* is the name given to a collection of ordered pairs. In Figure 2(b) we’ve plotted each of the ordered pairs in the relation $R$. This is called the *graph* of the relation $R$. 

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5 Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/
**Definition 1.** The graph of a relation is the collection of all ordered pairs of the relation. These are usually represented as points in a Cartesian coordinate system.

![Graph of a relation](image)

**Figure 2.** A mapping diagram and its graph.

In Figure 2(a), we’ve created a mapping diagram of the ordered pairs. Note that the domain object 3 is paired with two range elements, namely 1 and 4. Hence the relation $R$ is not a function. It is interesting to note that there are two points in the graph of $R$ in Figure 2(b) that have the same first coordinate, namely $(3, 1)$ and $(3, 4)$. This is a signal that the graph of the relation $R$ is not a function. In the next section we will discuss the Vertical Line Test, which will use this dual use of the first coordinate to determine when a relation is a not a function.

**Creating the Graph of a Function**

Some texts will speak of the graph of an equation, such as “Draw the graph of the equation $y = x^2$.” This instruction raises a number of difficulties.

- First, the instruction provides no direction to the reader; that is, what does the instruction mean? It’s not very helpful.

- Secondly, the instruction is incorrect. You don’t draw the graphs of equations. Rather, you draw the graphs of relations and/or functions. A graph is just another way of representing a function, a relation that pairs each element in its domain with exactly one element in its range.

So, what is the proper instruction? First, we will provide the formal definition of the graph of a function, then we will break it down by means of examples.
Definition 2. The graph of a function is the collection of all ordered pairs of the function. These are usually represented as points in a Cartesian coordinate system.

As an example, consider the function

\[ f = \{(1, 2), (2, 4), (3, 1), (4, 3)\}. \tag{3} \]

Readers will note that each object in the domain is paired with one and only one object in the range, as seen in the mapping diagram of Figure 3(a).

Thus, we have two representations of the function \( f \), the collection of ordered pairs (3), and the mapping diagram of in Figure 3(a). A third representation of the function \( f \) is the graph of the ordered pairs of the function, shown in the Cartesian plane in Figure 3(b).

![Figure 3. A mapping diagram and its graph.](image)

When the function is represented by an equation or formula, then we adjust our definition of its graph somewhat.

Definition 4. The graph of \( f \) is the set of all ordered pairs \((x, f(x))\) so that \( x \) is in the domain of \( f \). In symbols,

\[ \text{Graph of } f = \{(x, f(x)) : x \text{ is in the domain of } f \}. \]

This last definition is most easily explained by example. So, let’s define a function \( f \) that maps any real number \( x \) to the real number \( x^2 \); that is, let \( f(x) = x^2 \). Now, according to Definition 4, the graph of \( f \) is the set of all points \((x, f(x))\), such that \( x \) is in the domain of \( f \).

The way is now clear. We begin by creating a table of points \((x, f(x))\), where \( x \) is in the domain of the function \( f \) defined by \( f(x) = x^2 \). The choice of \( x \) is both subjective
and experimental, so we begin by choosing integer values of \( x \) between \(-3\) and \(3\). We then evaluate the function at each of these \( x \)-values (e.g., \( f(-3) = (-3)^2 = 9 \)). The results are shown in the table in Figure 4(a). We then plot the points in our table in the Cartesian plane as shown in Figure 4(b).

\[
\begin{array}{|c|c|c|}
\hline
x & f(x) = x^2 & (x, f(x)) \\
\hline
-3 & 9 & (-3, 9) \\
-2 & 4 & (-2, 4) \\
-1 & 1 & (-1, 1) \\
0 & 0 & (0, 0) \\
1 & 1 & (1, 1) \\
2 & 4 & (2, 4) \\
3 & 9 & (3, 9) \\
\hline
\end{array}
\]

**Figure 4.** Plotting pairs satisfying the functional relationship defined by the equation \( f(x) = x^2 \).

Although this is a good start, the graph in Figure 4(b) is far from complete. Definition 4 requires that we plot the ordered pairs \((x, f(x))\) for every value of \( x \) that is in the domain of \( f \). We’ve only plotted seven such points, so we’re not done. Let’s add more points to the graph of \( f \). We’ll evaluate the function at each of the \( x \)-values shown in the table in Figure 5(a), then plot the additional pairs \((x, f(x))\) from the table in the Cartesian plane, as shown in Figure 5(b).

\[
\begin{array}{|c|c|c|}
\hline
x & f(x) = x^2 & (x, f(x)) \\
\hline
-5/2 & 25/4 & (-5/2, 25/4) \\
-3/2 & 9/4 & (-3/2, 9/4) \\
-1/2 & 1/4 & (-1/2, 1/4) \\
1/2 & 1/4 & (1/2, 1/4) \\
3/2 & 9/4 & (3/2, 9/4) \\
5/2 & 25/4 & (5/2, 25/4) \\
\hline
\end{array}
\]

**Figure 5.** Plotting additional pairs \((x, f(x))\) defined by the equation \( f(x) = x^2 \).
We’re still not finished, because we’ve only plotted 13 pairs \((x, f(x))\), such that \(f(x) = x^2\). **Definition 4** requires that we plot the ordered pairs \((x, f(x))\) for every value of \(x\) in the domain of \(f\).

However, a pattern is certainly establishing itself, as seen in Figure 5(b). At some point, we need to “make a leap of faith,” and plot all ordered pairs \((x, f(x))\), such that \(x\) is in the domain of \(f\). This is done in Figure 6.

![Figure 6](image)

**Figure 6.** Plotting all pairs \((x, f(x))\) so that \(x\) is in the domain of \(f\).

There are several important points we need to make about the final result in Figure 6.

- When we draw a smooth curve, such as that shown in Figure 6, it is important to understand that this is a simply a shortcut for plotting all pairs \((x, f(x))\), where \(f(x) = x^2\) and \(x\) is in the domain of \(f\).\(^6\)
- It is important to understand that we are NOT “connecting the dots,” neither with a ruler nor with curved segments. Rather, the curve in Figure 6 is the result of plotting all of the individual pairs \((x, f(x))\).
- The “arrows” at each end of the curve have an important meaning. Much as the ellipsis at the end of the progression 2, 4, 6, \ldots mean “et-cetera,” the arrows at each end of the curve have a similar meaning. The arrow at the end of the left-half of the curve indicates that the graph continues opening upward and to the left, while the arrow at the end of the right-half of the curve indicates that the graph continues opening upward and to the right.

**Creating Graphs by Hand**

We’re going to look at several basic graphs, which we’ll create by employing the strategy used to create the graph of \(f(x) = x^2\). First, let’s summarize that process.

\(^6\) It would take too long to plot the individual pairs “one at a time.”
Summary 5. If a function is defined by an equation, you can create the graph of the function as follows.

1. Select several values of \( x \) in the domain of the function \( f \).
2. Use the selected values of \( x \) to create a table of pairs \((x, f(x))\) that satisfy the equation that defines the function \( f \).
3. Create a Cartesian coordinate system on a sheet of graph paper. Label and scale each axis, then plot the pairs \((x, f(x))\) from your table on your coordinate system.
4. If the plotted pairs \((x, f(x))\) provide enough of a pattern for you to intuit the shape of the graph of \( f \), make the “leap of faith” and plot all pairs that satisfy the equation defining \( f \) by drawing a smooth curve on your coordinate system. Of course, this curve should contain all previously plotted pairs.
5. If your plotted pairs do not provide enough of a pattern to determine the final shape of the graph of \( f \), then add more pairs to your table and plot them on your Cartesian coordinate system. Continue in this manner until you are confident in the shape of the graph of \( f \).

Let’s look at an example.

Example 6. Sketch the graph of the function defined by the equation \( f(x) = x^3 \).

We’ll start with \( x \)-values \(-2, -1, 0, 1, \) and \( 2 \), then use the equation \( f(x) = x^3 \) to determine pairs \((x, f(x))\) (e.g., \( f(-2) = (-2)^3 = -8 \)). These are listed in the table in Figure 7(a). We then plot the points from the table on a Cartesian coordinate system, as shown in Figure 7(b).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = x^3 )</th>
<th>( (x, f(x)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2)</td>
<td>(-8)</td>
<td>((-2, -8))</td>
</tr>
<tr>
<td>(-1)</td>
<td>(-1)</td>
<td>((-1, -1))</td>
</tr>
<tr>
<td>(0)</td>
<td>(0)</td>
<td>((0, 0))</td>
</tr>
<tr>
<td>(1)</td>
<td>(1)</td>
<td>((1, 1))</td>
</tr>
<tr>
<td>(2)</td>
<td>(8)</td>
<td>((2, 8))</td>
</tr>
</tbody>
</table>

Figure 7. Plotting pairs \((x, f(x))\) defined by the equation \( f(x) = x^3 \).
We’re a bit unsure of the shape of the graph of \( f \), so we’ll add a few more pairs to our table and plot them. This is shown in Figures 8(a) and (b).

\[
\begin{array}{|c|c|c|}
\hline
x & f(x) = x^3 & (x, f(x)) \\
\hline
-3/2 & -27/8 & (-3/2, -27/8) \\
-1/2 & -1/8 & (-1/2, -1/8) \\
1/2 & 1/8 & (1/2, 1/8) \\
3/2 & 27/8 & (3/2, 27/8) \\
\hline
\end{array}
\]

\[\text{(a)}\]

**Figure 8.** Plotting additional pairs \((x, f(x))\) defined by the equation \( f(x) = x^3 \).

The additional pairs fill in the shape of \( f \) in Figure 8(b) a bit better than those in Figure 7(b), enough so that we’re confident enough to make a “leap of faith” and draw the final shape of the graph of \( f(x) = x^3 \) in Figure 9.

\[\text{Figure 9.} \quad \text{The final graph of } f(x) = x^3.\]

Let’s look at another example.

**Example 7.** Sketch the graph of \( f(x) = \sqrt{x} \).

Again, we’ll start by selecting several values of \( x \) in the domain of \( f \). In this case, \( f(x) = \sqrt{x} \), and it’s not possible to take the square root of a negative number.\(^7\) Also,

\(^7\) Whenever you square a real number, the result is either positive or zero. Hence, the square root of a negative number cannot be a real number.
if we’re creating a table of pairs by hand, it’s good strategy to select known squares. Thus, we’ll use $x = 0, 1, 4,$ and $9$ for starters.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = \sqrt{x}$</th>
<th>$(x, f(x))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>(1, 1)</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>(4, 2)</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>(9, 3)</td>
</tr>
</tbody>
</table>

**Figure 10.** Plotting pairs $(x, f(x))$ defined by the equation $f(x) = \sqrt{x}$.

Some might be ready to make a “leap of faith” based on these initial results. Others might want to use a calculator to compute decimal approximations for additional square roots. The resulting pairs are shown in the table in **Figure 11(a)** and the additional pairs are plotted in **Figure 11(b)**.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = \sqrt{x}$</th>
<th>$(x, f(x))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.4</td>
<td>(2, 1.4)</td>
</tr>
<tr>
<td>3</td>
<td>1.7</td>
<td>(3, 1.7)</td>
</tr>
<tr>
<td>5</td>
<td>2.2</td>
<td>(5, 2.2)</td>
</tr>
<tr>
<td>6</td>
<td>2.4</td>
<td>(6, 2.4)</td>
</tr>
<tr>
<td>7</td>
<td>2.6</td>
<td>(7, 2.6)</td>
</tr>
<tr>
<td>8</td>
<td>2.8</td>
<td>(8, 2.8)</td>
</tr>
</tbody>
</table>

**Figure 11.** Plotting additional pairs $(x, f(x))$ defined by the equation $f(x) = \sqrt{x}$.

The pattern in **Figure 11(b)** is clear enough to make a “leap of faith” and complete the graph as shown in **Figure 12**.
Using the Table Feature of the Graphing Calculator

The TABLE feature on your graphing calculator can be of immense help when creating tables of points that satisfy the equation defining the function \( f \). Let’s look at an example.

**Example 8.** Sketch the graph of \( f(x) = |x| \).

Enter the function \( f(x) = |x| \) in the \( \text{Y}= \) menu as follows.\(^8\)

1. Press the \( \text{Y}= \) button on your calculator. This will open the \( \text{Y}= \) menu as shown in Figure 13(a). Use the arrow keys and the CLEAR button on your calculator to delete any existing functions.

2. Press the MATH button to open the menu shown in Figure 13(b).

3. Press the right-arrow on your calculator to select the NUM submenu as shown in Figure 13(c).

4. Select 1:abs(, then enter \( x \) and close the parentheses, as shown in Figure 13(d).

\(^8\) Readers will recall that the absolute value function takes a real number and makes it nonnegative. For example, \(|-3| = 3\), \(|0| = 0\), and \(|3| = 3\). We’ll have more to say about the absolute value function in Chapter 3.
We will now use the TABLE feature of the graphing calculator to help create a table of pairs \((x, f(x))\) satisfying the equation \(f(x) = |x|\). Proceed as follows.

1. Select 2nd TBLSET (i.e., push the 2nd button followed by TBLSET), which is located over the WINDOW button. Enter TblStart=-3, \(\Delta\text{Tbl} = 1\), and set the independent and dependent variables to Auto (this is done by highlighting Auto and pressing the Enter button), as shown in Figure 14(a).

2. Press 2nd TABLE, which is located above the GRAPH button, to produce the table of pairs \((x, f(x))\) shown in Figure 14(b).

We’ve plotted the pairs directly from the calculator onto a Cartesian coordinate system on graph paper in Figure 14(c).

![Figure 14. Creating a table with the TABLE feature of the graphing calculator.](image)

Based on what we see in Figure 14(c), we’re ready to make a “leap of faith” and draw the graph of \(f\) shown in Figure 15.

![Figure 15. The graph of \(f\) defined by \(f(x) = |x|\).](image)

Alternatively, or as a check, we can have the graphing calculator draw the graph for us. Push the ZOOM button, then select 6:ZStandard (shown in Figure 16(a)) to produce the graph shown in Figure 16(b).

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Adjusting the Viewing Window

In Example 8, we used the graphing calculator to draw the graph of the function defined by the equation \( f(x) = |x| \). For the functions we’ve encountered thus far, drawing their graphs using the graphing calculator is pretty trivial. Simply enter the equation in the \( Y= \) menu, then press the ZOOM button and select 6:ZStandard. However, if the graph of a function doesn’t fit (or even appear) in the “standard” viewing window, it can be quite challenging to find optimal view settings so that the important features of the graph are visible.

Indeed, as one might not even know what “important” features to look for, setting the viewing window is usually highly subjective and experimental by nature. Let’s look at some examples.

Example 9. Use a graphing calculator to sketch the graph of \( f(x) = 56 - x - x^2 \). Experiment with the WINDOW settings until you feel you have a viewing window that exhibits the important features of the graph.

First, start by entering the function in the \( Y= \) menu, as shown in Figure 17(a). The caret ` on the keyboard is used for exponents. Press the ZOOM button and select 6:ZStandard to produce the graph shown in Figure 17(b).

As the graph draws, observe that the graph rises from the bottom of the screen, leaves the top of the screen, then returns, falling from the top of the screen and leaving...
again at the bottom of the screen. This would indicate that there must be some sort of “turning point” that is not visible at the top of the screen.

Press the WINDOW button to reveal the “standard viewing window” settings shown in **Figure 18(a)**. The following legend explains each of the WINDOW parameters in **Figure 18(a)**.

\[
\begin{align*}
X_{\text{min}} &= \text{x-value of left edge of viewing window} \\
X_{\text{max}} &= \text{x-value of right edge of viewing window} \\
X_{\text{scl}} &= \text{x-axis tick increment} \\
Y_{\text{min}} &= \text{y-value of bottom edge of viewing window} \\
Y_{\text{max}} &= \text{y-value of top edge of viewing window} \\
Y_{\text{scl}} &= \text{y-axis tick increment}
\end{align*}
\]

It is easy to evaluate the function \( f(x) = 56 - x - x^2 \) at \( x = 0 \). Indeed, \( f(0) = 56 - 0 - 0^2 = 56 \). This indicates that the graph of \( f \) must pass through the point \((0, 56)\). This gives us a clue at how we should set the upper bound on our viewing window. Set \( Y_{\text{max}} = 60 \), as shown in **Figure 18(b)**, then press the GRAPH button to produce the graph and viewing window shown in **Figure 18(c)**.

![Figure 18. Changing the viewing window.](image)

Although the viewing window in **Figure 18(c)** shows the “turning point” of the graph of \( f \), we will make some additional changes to the window settings, as shown in **Figure 19(a)**. First, we “widen” the viewing window a bit, setting \( X_{\text{min}} = -15 \) and \( X_{\text{max}} = 15 \), then we set tick marks on the \( x \)-axis every 5 units with \( X_{\text{scl}} = 5 \). Next, to create a little room at the top of the screen, we set \( Y_{\text{max}} = 100 \), then we “balance” this setting with \( Y_{\text{min}} = -100 \). Finally, we set tick marks on the \( y \)-axis every 10 units with \( Y_{\text{scl}} = 10 \).

Push the GRAPH button to view the effects of these changes to the WINDOW parameters in **Figure 19(b)**. Note that these settings are highly subjective, and what one reader might find quite pleasing will not necessarily find favor with other readers.

However, what is important is the fact that we’ve captured the “important features” of the graph of \( f(x) = 56 - x - x^2 \). Note that this is a very controversial statement. If one is just beginning to learn about the graphs of functions, how is one to determine what are the “important features” of the graph? Unfortunately, the answer to this question is, “through experience.” Undoubtedly, this is a very frustrating phrase for readers to hear, but at least it’s truthful. The more graphs that you draw, the more
Section 2.2 The Graph of a Function

(a) (b)

Figure 19. Improving the WINDOW settings.

you will learn how to look for “turning points,” “end-behavior,” “$x$- and $y$-intercepts,” and the like.

For example, how do we know that the WINDOW settings in Figure 19(a) determine a viewing window (Figure 19(b)) that reveals all “important features” of the graph? The answer at this point is, “we don’t, not without further experiment.” For example, the careful reader might want to try the window settings $X_{\text{min}}=-50$, $X_{\text{max}}=50$, $X_{\text{scl}}=10$, $Y_{\text{min}}=-500$, $Y_{\text{max}}=500$, and $Y_{\text{scl}}=100$ to see if any unexpected behavior crops up.

Let’s look at one last example.

Example 10. Sketch the graph of the function $f$ defined by the equation $f(x) = x^4 + 9x^3 - 117x^2 - 265x + 2100$.

Load the function into the $Y=$ menu (shown in Figure 20(a)) and select 6:ZStandard to produce the graph shown in Figure 20(b).

As the graph draws, observe that it rises form the bottom of the viewing window, leaves the top of the viewing window, then returns to fall off the bottom of the viewing window, then returns again and rises off the top of the viewing window.

We notice that $f(0) = 2100$, so we’ll need to set the top of the viewing window to that value or higher. With this thought in mind, we’ll set $Y_{\text{max}}=3000$, then set $Y_{\text{min}}=-3000$ for balance, then to avoid a million little tick marks, we’ll set $Y_{\text{scl}}=1000$, all shown in Figure 21(a). Pressing the GRAPH button then produces the image shown in Figure 21(b).
Does it appear that we have all of the “important features” of this graph displayed in our viewing window? Note that we did not experiment very much. Perhaps we should try expanding the window a bit more to see if we have missed any important behavior. With that thought in mind, we set $X_{\text{min}}=-20$, $X_{\text{max}}=20$, and to avoid a ton of tick marks, $X_{\text{scl}}=5$, as shown in Figure 22(a). Pushing the GRAP button produces the image in Figure 22(b).

Note that the viewing window in Figure 22(b) reveals behavior not seen in the viewing window of Figure 21(b). If we had not experimented further, if we had not expanded the viewing window, we would not have seen this new behavior. This is an important lesson.

Note that one of the “turning points” of the graph in Figure 22(b) lies off the bottom of the viewing window. We’ll make one more adjustment to include this important feature. Set $Y_{\text{min}}=-10000$, $Y_{\text{max}}=10000$, and $Y_{\text{scl}}=5000$, as shown in Figure 23(a), then push the GRAP button to produce the image shown in Figure 23(b).

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The graph in Figure 23(b) shows all of the “important features” of the graph of \( f \), but the careful reader will continue to experiment, expanding the viewing window to ascertain the truth of this statement.
2.2 Exercises

Perform each of the following tasks for the functions defined by the equations in Exercises 1-8.

i. Set up a table of points that satisfy the given equation. Please place this table of points next to your graph on your graph paper.

ii. Set up a coordinate system on a sheet of graph paper. Label and scale each axis, then plot each of the points from your table on your coordinate system.

iii. If you are confident that you “see” the shape of the graph, make a “leap of faith” and plot all pairs that satisfy the given equation by drawing a smooth curve (free-hand) on your coordinate system that contains all previously plotted points (use a ruler only if the graph of the equation is a line). If you are not confident that you “see” the shape of the graph, then add more points to your table, plot them on your coordinate system, and see if this helps. Continue this process until you “see” the shape of the graph and can fill in the rest of the points that satisfy the equation by drawing a smooth curve (or line) on your coordinate system.

1. \( f(x) = 2x + 1 \)
2. \( f(x) = 1 - x \)
3. \( f(x) = 3 - \frac{1}{2} x \)
4. \( f(x) = -1 + \frac{1}{2} x \)
5. \( f(x) = x^2 - 2 \)
6. \( f(x) = 4 - x^2 \)
7. \( f(x) = \frac{1}{2} x^2 - 6 \)
8. \( f(x) = 8 - \frac{1}{2} x^2 \)

Perform each of the following tasks for the functions Exercises 9-10.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis.

ii. Use the table feature of your graphing calculator to evaluate the function at the given values of \( x \). Record these results in a table next to your coordinate system on your graph paper.

iii. Plot the points in the table on your coordinate system then use them to draw the graph of the given function. Label the graph with its equation.

9. \( f(x) = \sqrt{x - 4} \) at \( x = 4, 5, 6, 7, 8, 9, \) and 10.
10. \( f(x) = \sqrt{4 - x} \) at \( x = -10, -8, -6, -4, -2, 0, 2, \) and 4.

In Exercises 11-14, the graph of the given function is a parabola, a graph that has a “U-shape.” A parabola has only one turning point. For each exercise, perform the following tasks.

i. Load the equation into the \( Y= \) menu of your graphing calculator. Adjust the \textsc{window} parameters so that the “turning point” (actually called the vertex) is visible in the viewing window.

ii. Make a reasonable copy of the image in the viewing window on your home-
work paper. Draw all lines with a ruler (including the axes), but draw curves freehand. Label and scale each axis with xmin, xmax, ymin, and ymax. Label the graph with its equation.

11. \( f(x) = x^2 - x - 30 \)
12. \( f(x) = 24 - 2x - x^2 \)
13. \( f(x) = 11 + 10x - x^2 \)
14. \( f(x) = x^2 + 11x - 12 \)

Each of the equations in Exercises 15-18 are called “cubic polynomials.” Each equation has been carefully chosen so that its graph has exactly two “turning points.” For each exercise, perform each of the following tasks.

i. Load the equation into the Y= menu of your graphing calculator and adjust the WINDOW parameters so that both “turning points” are visible in the viewing window.

ii. Make a reasonable copy of the graph in the viewing window on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax, then label the graph with its equation. Remember to draw all lines with a ruler.

15. \( f(x) = x^3 - 2x^2 - 29x + 30 \)
16. \( f(x) = -x^3 + 2x^2 + 19x - 20 \)
17. \( f(x) = x^3 + 8x^2 - 53x - 60 \)
18. \( f(x) = -x^3 + 16x^2 - 43x - 60 \)

Perform each of the following tasks for the equations in Exercises 19-22.

i. Load the equation into the Y= menu. Adjust the WINDOW parameters until you think all important behavior (“turning points,” etc.) is visible in the viewing window. Note: This is more difficult than it sounds, particularly when we have no advance notion of what the graph might look like. However, experiment with several settings until you “discover” the settings that exhibit the most important behavior.

ii. Copy the image on the screen onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label the graph with its equation.

19. \( f(x) = 2x^2 - x - 465 \)
20. \( f(x) = x^3 - 24x^2 + 65x + 1050 \)
21. \( f(x) = x^4 - 2x^3 - 168x^2 + 288x + 3456 \)
22. \( f(x) = -x^4 - 3x^3 + 141x^2 + 523x - 660 \)
2.2 Answers

1. \[ f(x) = 2x + 1 \]

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<th>(f(x))</th>
<th>((x, f(x)))</th>
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</tr>
<tr>
<td>1</td>
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3. \[ f(x) = 3 - x/2 \]

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<td>2</td>
<td>(2, 2)</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>(4, 1)</td>
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</table>

5. \[ f(x) = x^2 - 2 \]

<table>
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<th>((x, f(x)))</th>
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<tr>
<td>3</td>
<td>7</td>
<td>(3, 7)</td>
</tr>
</tbody>
</table>

7. \[ f(x) = x^2/2 - 6 \]

<table>
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<th>(f(x))</th>
<th>((x, f(x)))</th>
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<tr>
<td>4</td>
<td>2</td>
<td>(4, 2)</td>
</tr>
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</table>
Chapter 2 Functions

11. \( f(x) = x^2 - x - 30 \)

13. \( f(x) = 11 + 10x - x^2 \)

15. \( f(x) = x^3 - 2x^2 - 29x + 30 \)
17. $f(x)=x^3+8x^2-53x-60$

19. $f(x)=2x^2-x-465$

21. $f(x)=x^4-2x^3-168x^2+288x+3456$
2.3 Interpreting the Graph of a Function

In the previous section, we began with a function and then drew the graph of the given function. In this section, we will start with the graph of a function, then make a number of interpretations based on the given graph: function evaluations, the domain and range of the function, and solving equations and inequalities.

**The Vertical Line Test**

Consider the graph of the relation $R$ shown in Figure 1(a). Recall that we earlier defined a relation as a set of ordered pairs. Surely, the graph shown in Figure 1(a) is a set of ordered pairs. Indeed, it is an infinite set of ordered pairs, so many that the graph is a solid curve.

In Figure 1(b), note that we can draw a vertical line that cuts the graph more than once. In Figure 1(b), we’ve drawn a vertical line that cuts the graph in two places, once at $(x, y_1)$, then again at $(x, y_2)$, as shown in Figure 1(c). This means that the domain object $x$ is paired with two different range objects, namely $y_1$ and $y_2$, so relation $R$ is not a function.

Recall the definition of a function.

**Definition 1.** A relation is a **function** if and only if each object in its domain is paired with one and only one object in its range.

Consider the mapping diagram in Figure 2, where we’ve used arrows to indicate the ordered pairs $(x, y_1)$ and $(x, y_2)$ in Figure 1(c). Note that $x$, an object in the domain of $R$, is mapped to two objects in the range of $R$, namely $y_1$ and $y_2$. Hence, the relation $R$ is not a function.
Figure 2. A mapping diagram representing the points \((x, y_1)\) and \((x, y_2)\) in Figure 1(c).

This discussion leads to the following result, called the vertical line test for functions.

**The Vertical Line Test.** If any vertical line cuts the graph of a relation more than once, then the relation is **not** a function.

Hence, the circle pictured in Figure 3(a) is a relation, but it is **not** the graph of a function. It is possible to cut the graph of the circle more than once with a vertical line, as shown in Figure 3(a). On the other hand, the parabola shown in Figure 3(b) is the graph of a function, because no vertical line will cut the graph more than once.

Figure 3. Use the vertical line test to determine if the graph is the graph of a function.

**Reading the Graph for Function Values**

We know that the graph of \(f\) pictured in Figure 4 is the graph of a function. We know this because no vertical line will cut the graph of \(f\) more than once.

We earlier defined the graph of \(f\) as the set of all ordered pairs \((x, f(x))\), so that \(x\) is in the domain of \(f\). Consequently, if we select a point \(P\) on the graph of \(f\), as in Figure 4(a), we label the point \(P(x, f(x))\). However, we can also label this point as \(P(x, y)\), as shown in Figure 4(b). This leads to a new interpretation of \(f(x)\) as the \(y\)-value of the point \(P\). That is, \(f(x)\) is the \(y\)-value that is paired with \(x\).\(^{11}\)

\(^{11}\) Of course, if the axes were labeled \(A\) and \(t\), then there would be a similar interpretation based on the variables \(A\) and \(t\).
Section 2.3 Interpreting the Graph of a Function

Figure 4. Reading the graph of a function.

Definition 2. \( f(x) \) is the \( y \)-value that is paired with \( x \).

Two more comments are in order. In Figure 4(a), we select a point \( P \) on the graph of \( f \).

1. To find the \( x \)-value of the point \( P \), we must project the point \( P \) onto the \( x \)-axis.

2. To find \( f(x) \), the value of \( y \) that is paired with \( x \), we must project the point \( P \) onto the \( y \)-axis.

Let’s look at an example.

Example 3. Given the graph of \( f \) in Figure 5(a), find \( f(4) \).

First, note that the graph of \( f \) represents a function. No vertical line will cut the graph of \( f \) more than once.

Because \( f(4) \) represents the \( y \)-value that is paired with an \( x \)-value of 4, we first locate 4 on the \( x \)-axis, as shown in Figure 5(b). We then draw a vertical arrow until we intercept the graph of \( f \) at the point \( P(4, f(4)) \). Finally, we draw a horizontal arrow...
from the point $P$ until we intercept the $y$-axis. The projection of the point $P$ onto the $y$-axis is the value of $f(4)$.

Because we have a grid that shows a scale on each axis, we can approximate the value of $f(4)$. It would appear that the $y$-value of point $P$ is approximately 4. Thus, $f(4) \approx 4$.

Let’s look at another example.

**Example 4.** Given the graph of $f$ in Figure 6(a), find $f(5)$.

![Figure 6](image-url)  

**Figure 6.** Finding the value of $f(5)$.

First, note that the graph of $f$ represents a function. No vertical line will cut the graph of $f$ more than once.

Because $f(5)$ represents the $y$-value that is paired with an $x$-value of 5, we first locate 5 on the $x$-axis, as shown in Figure 6(b). We then draw a vertical arrow until we intercept the graph of $f$ at the point $P(5, f(5))$. Finally, we draw a horizontal arrow from the point $P$ until we intercept the $y$-axis. The projection of the point $P$ onto the $y$-axis is the value of $f(5)$.

Because we have a grid that shows a scale on each axis, we can approximate the value of $f(5)$. It would appear that the $y$-value of point $P$ is approximately 6. Thus, $f(5) \approx 6$.

Let’s reverse the interpretation in another example.

**Example 5.** Given the graph of $f$ in Figure 7(a), for what value of $x$ does $f(x) = -4$?

Again, the graph in Figure 7 passes the vertical line test and represents the graph of a function.

This time, in the equation $f(x) = -4$, we’re given a $y$-value equal to $-4$. Consequently, we must reverse the process used in Example 3 and Example 4. We first locate the $y$-value $-4$ on the $y$-axis, then draw a horizontal arrow until we intercept...
Section 2.3 Interpreting the Graph of a Function

Figure 7. Finding \( x \) so that \( f(x) = -4 \).

the graph of \( f \) at \( P \), as shown in Figure 7(b). Finally, we draw a vertical arrow from the point \( P \) until we intercept the \( x \)-axis. The projection of the point \( P \) onto the \( x \)-axis is the solution of \( f(x) = -4 \).

Because we have a grid that shows a scale on each axis, we can approximate the \( x \)-value of the point \( P \). It seems that \( x \approx 5 \). Thus, we label the point \( P(5, f(5)) \), and the solution of \( f(x) = -4 \) is approximately \( x \approx 5 \).

This solution can easily be checked by computing \( f(5) \). Simply start with 5 on the \( x \)-axis, then reverse the order of the arrows shown in Figure 7(b). You should wind up at \(-4\) on the \( y \)-axis, demonstrating that \( f(5) = -4 \).

The Domain and Range of a Function

We can use the graph of a function to determine its domain and range. For example, consider the graph of the function shown in Figure 8(a).

Figure 8. Determining the domain of a function from its graph.

Note that no vertical line will cut the graph of \( f \) more than once, so the graph of \( f \) represents a function.
To determine the domain, we must collect the \( x \)-values (first coordinates) of every point on the graph of \( f \). In Figure 8(b), we’ve selected a point \( P \) on the graph of \( f \), which we then project onto the \( x \)-axis. The image of this projection is the point \( Q \), and the \( x \)-value of the point \( Q \) is an element in the domain of \( f \).

Think of the projection shown in Figure 8(b) in the following manner. Imagine a light source above the point \( P \). The point \( P \) blocks out the light and its shadow falls onto the \( x \)-axis at the point \( Q \). That is, think of point \( Q \) as the “shadow” that the point \( P \) produces when it is projected vertically onto the \( x \)-axis.

Now, to find the domain of the function \( f \), we must project each point on the graph of \( f \) onto the \( x \)-axis. Here’s the question: if we project each point on the graph of \( f \) onto the \( x \)-axis, what part of the \( x \)-axis will “lie in shadow” when the process is complete? The answer is shown in Figure 8(c).

In Figure 8(c), note that the “shadow” created by projecting each point on the graph of \( f \) onto the \( x \)-axis is shaded in red (a thicker line if you are viewing this in black and white). This collection of \( x \)-values is the domain of the function \( f \). There are three critical points that we need to make about the “shadow” on the \( x \)-axis in Figure 8(c).

1. All points lying between \( x = -3 \) and \( x = 4 \) have been shaded on the \( x \)-axis in red.

2. The left endpoint of the graph of \( f \) is an open circle. This indicates that there is no point plotted at this endpoint. Consequently, there is no point to project onto the \( x \)-axis, and this explains the open circle at the left end of our “shadow” on the \( x \)-axis.

3. On the other hand, the right endpoint of the graph of \( f \) is a filled endpoint. This indicates that this is a plotted point and part of the graph of \( f \). Consequently, when this point is projected onto the \( x \)-axis, a shadow falls at \( x = 4 \). This explains the filled endpoint at the right end of our “shadow” on the \( x \)-axis.

We can describe the \( x \)-values of the “shadow” on the \( x \)-axis using set-builder notation.

\[
\text{Domain of } f = \{ x : -3 < x \leq 4 \}.
\]

Note that we don’t include \(-3\) in this description because the left end of the shadow on the \( x \)-axis is an empty circle. Note that we do include 4 in this description because the right end of the shadow on the \( x \)-axis is a filled circle.

We can also describe the \( x \)-values of the “shadow” on the \( x \)-axis using interval notation.

\[
\text{Domain of } f = (-3, 4]
\]

We remind our readers that the parenthesis on the left means that we are not including \(-3\), while the bracket on the right means that we are including 4.

To find the range of the function, picture again the graph of \( f \) shown in Figure 9(a). Proceed in a similar manner, only this time project points on the graph of \( f \) onto the \( y \)-axis, as shown in Figures 9(b) and (c).
Section 2.3 Interpreting the Graph of a Function

Figure 9. Determining the range of a function from its graph.

Note which part of the $y$-axis “lies in shadow” once we’ve projected all points on the graph of $f$ onto the $y$-axis.

1. All points lying between $y = -2$ and $y = 4$ have been shaded on the $y$-axis in red (a thicker line style if you are viewing this in black and white).

2. The left endpoint of the graph of $f$ is an empty circle, so there is no point to project onto the $y$-axis. Consequently, there is no “shadow” at $y = -2$ on the $y$-axis and the point is left unshaded (an empty circle).

3. The right endpoint of the graph of $f$ is a filled circle, so there is a “shadow” at $y = 4$ on the $y$-axis and this point is shaded (a filled circle).

We can now easily describe the range in both set-builder and interval notation.

Range of $f = (-2, 4] = \{y : -2 < y \leq 4\}$

Let’s look at another example.

Example 6. Use set-builder and interval notation to describe the domain and range of the function represented by the graph in Figure 10(a).

Figure 10. Determining the domain from the graph of $f$. 
To determine the domain of $f$, project each point on the graph of $f$ onto the $x$-axis. This projection is indicated by the “shadow” on the $x$-axis in Figure 10(b). Two important points need to be made about this “shadow” or projection.

1. The left endpoint of the graph of $f$ is empty (indicated by the open circle), so it has no projection onto the $x$-axis. This is indicated by an open circle at the left end (at $x = -4$) of the “shadow” or projection on the $x$-axis.

2. The arrowhead on the right end of the graph of $f$ indicates that the graph of $f$ continues downward and to the right indefinitely. Consequently, the projection onto the $x$-axis is a shadow that moves indefinitely to the right. This is indicated by an arrowhead at the right end of the “shadow” or projection on the $x$-axis.

Consequently, the domain of $f$ is the collection of $x$-values represented by the “shadow” or projection onto the $x$-axis. Note that all $x$-values to the right of $x = -4$ are shaded on the $x$-axis. Consequently,

$$\text{Domain of } f = (-4, \infty) = \{x : x > -4\}.$$  

To find the range, we must project each point on the graph of $f$ (redrawn in Figure 11(a)) onto the $y$-axis. The projection is indicated by a “shadow” or projection on the $y$-axis, as seen in Figure 11(b). Two important points need to be made about this “shadow” or projection.

1. The left endpoint of the graph of $f$ is empty (indicated by an open circle), so it has no projection onto the $y$-axis. This is indicated by an open circle at the top end (at $y = 3$) of the “shadow” on the $y$-axis.

2. The arrowhead on the right end of the graph of $f$ indicates that the graph of $f$ continues downward and to the right indefinitely. Consequently, the projection of the graph of $f$ onto the $y$-axis is a shadow that moves indefinitely downward. In Figure 11(b), note how projections of points on the graph of $f$ not visible in the viewing window come in from the lower right corner and cast “shadows” on the $y$-axis.

Figure 11. Determining the range from the graph of $f$. 

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Consequently, the range of \( f \) is the collection of \( y \)-values shaded on the \( y \)-axis of the coordinate system shown in Figure 11(b). Note that all \( y \)-values lower than \( y = 3 \) are shaded on the \( y \)-axis. Thus, the range of \( f \) is

\[
\text{Range of } f = (-\infty, 3) = \{ y : y < 3 \}.
\]

Let’s look at another example.

**Example 7.** Use set-builder and interval notation to describe the domain and range of the function represented by the graph in Figure 12(a).

To determine the domain of \( f \), we must project all points on the graph of \( f \) onto the \( x \)-axis. This projection is indicated by the red “shadow” (or thicker line style if you are viewing this in black and white) shown on the \( x \)-axis in Figure 12(b). Two important points need to be made about this “shadow” or projection.

1. The arrow at the end of the left half of the graph of \( f \) in Figure 12(a) indicates that this half of the graph of \( f \) opens indefinitely to the left and upward. Consequently, when the points on the left half of the graph of \( f \) are projected onto the \( x \)-axis, the “shadow” or projection extends indefinitely to the left. Note how points on the graph that fall outside the viewing window come in from the upper left corner and cast “shadows” on the \( x \)-axis.

2. The arrow at the end of the right half of the graph of \( f \) in Figure 12(a) indicates that this half of the graph of \( f \) opens indefinitely to the right and upward. Consequently, when the points on this half of the graph of \( f \) are projected onto the \( x \)-axis, the “shadow” or projection extends indefinitely to the right.

Consequently, the entire \( x \)-axis lies in “shadow,” making the domain of \( f \) to be

\[
\text{Domain of } f = (-\infty, \infty) = \{ x : x \in \mathbb{R} \}.
\]
To determine the range of \( f \), we must project all points on the graph of \( f \) onto the \( y \)-axis. This projection is indicated by the red “shadow” (or thicker line if you are viewing this in black and white) shown on the \( y \)-axis in Figure 13(b). Two important points need to made about this “shadow” or projection.

1. The graph of \( f \) passes through the origin (the point \((0,0)\)). This is the lowest point on the graph and hence its shadow is the endpoint on the low end of the shaded region on the \( y \)-axis.

2. The arrows at the end of each half of the graph of \( f \) indicate that the graph opens upward indefinitely. Hence, when points on the graph of \( f \) are projected onto the \( y \)-axis, the “shadow” or projection extends upward indefinitely. This is indicated by an arrow on the upper end of the “shadow” on the \( y \)-axis.

Consequently, all points on the \( y \)-axis above and including the point at the origin “lie in shadow.” Thus, the range of \( f \) is

\[
\text{Range of } f = [0, \infty) = \{y : y \geq 0\}.
\]

**Using a Graphing Calculator to Determine Domain and Range**

We’ve learned how to find the domain and range of a function by looking at its graph. Therefore, if we define a function by means of an expression, such as \( f(x) = \sqrt{4 - x} \), then we should be able to capture the domain and range of \( f \) from its graph, provided, of course, that we can draw the graph of \( f \). We’ll find the graphing calculator will be a handy tool for this exercise.

**Example 8.** Use set-builder and interval notation to describe the domain and range of the function defined by the rule

\[
f(x) = \sqrt{4 - x}.
\]

Load the expression defining \( f \) into the \( Y= \) menu, as shown in Figure 14(a). Select 6:ZStandard from the ZOOM menu to produce the graph of \( f \) shown in Figure 14(b).
Section 2.3 Interpreting the Graph of a Function

Figure 14. Sketching the graph of \( f(x) = \sqrt{4-x} \).

Copy the image in Figure 14(b) onto a sheet of graph paper. Label and scale each axis with the WINDOW parameters \( x_{\text{min}}, x_{\text{max}}, y_{\text{min}}, \) and \( y_{\text{max}} \), as shown in Figure 15(a).

Figure 15. Capturing the domain of \( f(x) = \sqrt{4-x} \) from its graph.

Next, project each point on the graph of \( f \) onto the \( x \)-axis, as shown in Figure 15(b). Note that we’ve made two assumptions about the graph of \( f \).

1. At the left end of the graph in Figures 14(b) and 15(b), we assume that the graph of \( f \) continues upward and to the left indefinitely. Hence, the “shadow” or projection onto the \( x \)-axis will move indefinitely to the left. This is indicated by attaching an arrowhead to the left-hand end of the region that “lies in shadow” on the \( x \)-axis, as shown in Figure 15(b).

2. We also assume that the right end of the graph ends at the point \((4,0)\). This accounts for the “filled dot” when this point on the graph of \( f \) is projected onto the \( x \)-axis.

Note that the “shadow” or projection onto the \( x \) axis in Figure 15(b) includes all values of \( x \) less than or equal to 4. Thus, the domain of \( f \) is

\[
\text{Domain of } f = (-\infty, 4] = \{ x : x \leq 4 \}.
\]

We can intuit this result by considering the expression that defines \( f \). That is, consider the rule or definition

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\[ f(x) = \sqrt{4-x}. \]

Recall that we earlier defined the domain of \( f \) as the set of “permissible” \( x \)-values. In this case, it is impossible to take the square root of a negative number, so we must be careful selecting the \( x \)-values we use in this rule. Note that \( x = 4 \) is allowable, as

\[ f(0) = \sqrt{4-4} = \sqrt{0} = 0. \]

However, numbers larger than 4 cannot be used in this rule. For example, consider what happens when we attempt to use \( x = 5 \).

\[ f(x) = \sqrt{4-5} = \sqrt{-1} \]

This result is not a real number, so 5 is not in the domain of \( f \).

On the other hand, if we try \( x \)-values that are smaller than 4, such as \( x = 3 \),

\[ f(3) = \sqrt{4-3} = \sqrt{1} = 1. \]

We’ll leave it to our readers to test other values of \( x \) that are less than 4. They will also produce real answers when they are input into the rule \( f(x) = \sqrt{4-x} \). Note that this also verifies our earlier conjecture that the “shadow” or projection shown in Figure 15(b) continues indefinitely to the left.

Instead of “guessing and checking,” we can speed up the analysis of the domain of \( f(x) = \sqrt{4-x} \) by noting that the expression under the radical must not be a negative number. Hence, \( 4-x \) must either be greater than or equal to zero. This argument produces an inequality that is easily solved for \( x \).

\[
\begin{align*}
4-x & \geq 0 \\
-x & \geq -4 \\
x & \leq 4
\end{align*}
\]

This last result verifies that the domain of \( f \) is all values of \( x \) that are less than or equal to 4, which is in complete agreement with the “shadow” or projection onto the \( x \)-axis shown in Figure 15(b).

To determine the range of \( f \), we must project each point on the graph of \( f \) onto the \( y \)-axis, as shown in Figure 16(b).

Again, we make two assumptions about the graph of \( f \).

1. At the left-end of the graph of \( f(x) = \sqrt{4-x} \) in Figures 14(b) and 16(b), we assume that the graph of \( f \) continues upward and to the left indefinitely. Thus, when points on the graph of \( f \) are projected onto the \( y \)-axis, there will be projections coming from the upper left from points on the graph of \( f \) that are not visible in the viewing window selected in Figure 14(b). Hence, the “shadow” or projection on the \( y \)-axis shown in Figure 16(b) continues upward indefinitely. This is indicated with a arrowhead at the upper end of the “shadow” on the \( y \)-axis in Figure 16(b).

2. Again, we assume that the right end of the graph of \( f \) ends at the point \((4,0)\). The projection of this point onto the \( y \)-axis produces the “filled” endpoint at the origin shown in Figure 16(b).
Figure 16. Determining the range of \( f(x) = \sqrt{4 - x} \) from its graph.

Note that the “shadow” or projection onto the \( y \)-axis in Figure 16(b) includes all values of \( y \) that are greater than or equal to zero. Hence,

\[
\text{Range of } f = [0, \infty) = \{ y : y \geq 0 \}.
\]
2.3 Exercises

For Exercises 1-6, perform each of the following tasks.

i. Make a copy of the graph on a sheet of graph paper and apply the vertical line test.

ii. Write a complete sentence stating whether or not the graph represents a function. Explain the reason for your response.

1. 

2. 

3. 

4. 

5. 

6. 

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In Exercises 7-12, perform each of the following tasks.

i. Make an exact copy of the graph of the function $f$ on a sheet of graph paper. Label and scale each axis. Remember to draw all lines with a ruler.

ii. Use the technique of Examples 3 and 4 in the narrative to evaluate the function at the given value. Draw and label the arrows as shown in Figures 4 and 5 in the narrative.

7. Use the graph of $f$ to determine $f(2)$.

8. Use the graph of $f$ to determine $f(3)$.

9. Use the graph of $f$ to determine $f(-2)$.

10. Use the graph of $f$ to determine $f(1)$.
11. Use the graph of $f$ to determine $f(1)$.

12. Use the graph of $f$ to determine $f(-2)$.

13. Use the graph of $f$ to solve the equation $f(x) = -2$.

14. Use the graph of $f$ to solve the equation $f(x) = 1$.

15. Use the graph of $f$ to solve the equation $f(x) = 2$.

In Exercises 13-18, perform each of the following tasks.

i. Make an exact copy of the graph of the function $f$ on a sheet of graph paper. Label and scale each axis. Remember to draw all lines with a ruler.

ii. Use the technique of Example 5 in the narrative to find the value of $x$ that maps onto the given value. Draw and label the arrows as shown in Figure 6 in the narrative.
16. Use the graph of $f$ to solve the equation $f(x) = -2$.

17. Use the graph of $f$ to solve the equation $f(x) = 2$.

18. Use the graph of $f$ to solve the equation $f(x) = -3$.

In the Exercises 19-22, perform each of the following tasks.

i. Make a copy of the graph of $f$ on a sheet of graph paper. Label and scale each axis.

ii. Using a different colored pen or pencil, project each point on the graph of $f$ onto the $x$-axis. Shade the resulting domain on the $x$-axis.

iii. Use both set-builder and interval notation to describe the domain.
In Exercises 23-26, perform each of the following tasks.

i. Make a copy of the graph of $f$ on a sheet of graph paper. Label and scale each axis.

ii. Using a different colored pen or pencil, project each point on the graph of $f$ onto the $y$-axis. Shade the resulting range on the $y$-axis.

iii. Use both set-builder and interval notation to describe the range.
In Exercises 27-30, perform each of the following tasks.

i. Use your graphing calculator to draw the graph of the given function. Make a reasonably accurate copy of the image in your viewing screen on your homework paper. Label and scale each axis with the WINDOW parameters xmin, xmax, ymin, and ymax. Label the graph with its equation.

ii. Using a colored pencil, project each point on the graph onto the x-axis; i.e., shade the domain on the x-axis. Use interval and set-builder notation to describe the domain.

iii. Use a purely algebraic technique, as demonstrated in Example 8 in the narrative, to find the domain. Compare this result with that found in part (ii).

iv. Using a different colored pencil, project each point on the graph onto the y-axis; i.e., shade the range on the y-axis. Use interval and set-builder notation to describe the range.

27. \( f(x) = \sqrt{x + 5} \).

28. \( f(x) = \sqrt{5 - x} \).

29. \( f(x) = -\sqrt{4 - x} \).

30. \( f(x) = -\sqrt{x + 4} \).
2.3 Answers

1. Note that in the figure below a vertical line cuts the graph more than once. Therefore, the graph does not represent the graph of a function.

3. No vertical line cuts the graph more than once (see figure below). Therefore, the graph represents a function.

5. Note that in the figure below a vertical line cuts the graph more than once. Therefore, the graph does not represent the graph of a function.

7. $f(2) = -1$

9. $f(-2) = 1$
11. \( f(1) = 3 \)

13. The solution of \( f(x) = -2 \) is \( x = -3 \).

15. The solution of \( f(x) = 2 \) is \( x = -2 \).

17. The solution of \( f(x) = 2 \) is \( x = -1 \).

19. \( \{x : x > -3\} = (-3, \infty) \)

21. \( \{x : x < 0\} = (-\infty, 0) \)
23. \( \{y : y < 1\} = (-\infty, 1) \)

25. \( \{y : y > -2\} = (-2, \infty) \)

27. Domain = \([-5, \infty)\) 
\(= \{x : x \geq -5\} \)

29. Domain = \((-\infty, 4] = \{x : x \leq 4\} \)

Range = \(\{y : y \geq 0\} = [0, \infty) \)

Range = \(\{y : y \leq 0\} = (-\infty, 0]\)
2.4 Solving Equations and Inequalities by Graphing

Our emphasis in the chapter has been on functions and the interpretation of their graphs. In this section, we continue in that vein and turn our exploration to the solution of equations and inequalities by graphing. The equations will have the form \( f(x) = g(x) \), and the inequalities will have form \( f(x) < g(x) \) and/or \( f(x) > g(x) \).

You might wonder why we have failed to mention inequalities having the form \( f(x) \leq g(x) \) and \( f(x) \geq g(x) \). The reason for this omission is the fact that the solution of the inequality \( f(x) \leq g(x) \) is simply the union of the solutions of \( f(x) = g(x) \) and \( f(x) < g(x) \). After all, \( \leq \) is pronounced “less than or equal.” Similar comments are in order for the inequality \( f(x) \geq g(x) \).

We will begin by comparing the function values of two functions \( f \) and \( g \) at various values of \( x \) in their domains.

**Comparing Functions**

Suppose that we evaluate two functions \( f \) and \( g \) at a particular value of \( x \). One of three outcomes is possible. Either

\[
\text{either } \quad f(x) = g(x), \quad \text{or} \quad f(x) > g(x), \quad \text{or} \quad f(x) < g(x).
\]

It’s pretty straightforward to compare two function values at a particular value if rules are given for each function.

**Example 1.** Given \( f(x) = x^2 \) and \( g(x) = 2x + 3 \), compare the functions at \( x = -2, 0, \text{ and } 3 \).

Simple calculations reveal the relations.

- At \( x = -2 \),
  \[
  f(-2) = (-2)^2 = 4 \quad \text{and} \quad g(-2) = 2(-2) + 3 = -1,
  \]
  so clearly, \( f(-2) > g(-2) \).
- At \( x = 0 \),
  \[
  f(0) = (0)^2 = 0 \quad \text{and} \quad g(0) = 2(0) + 3 = 3,
  \]
  so clearly, \( f(0) < g(0) \).
- Finally, at \( x = 3 \),
  \[
  f(3) = (3)^2 = 9 \quad \text{and} \quad g(3) = 2(3) + 3 = 9,
  \]
  so clearly, \( f(3) = g(3) \).

We can also compare function values at a particular value of \( x \) by examining the graphs of the functions. For example, consider the graphs of two functions \( f \) and \( g \) in Figure 1.

[Figure 1: Graphs of functions f and g]

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Next, suppose that we draw a dashed vertical line through the point of intersection of the graphs of $f$ and $g$, then select a value of $x$ that lies to the left of the dashed vertical line, as shown in Figure 2(a). Because the graph of $f$ lies above the graph of $g$ for all values of $x$ that lie to the left of the dashed vertical line, it will be the case that $f(x) > g(x)$ for all such $x$ (see Figure 2(a)).

On the other hand, the graph of $f$ lies below the graph of $g$ for all values of $x$ that lie to the right of the dashed vertical line. Hence, for all such $x$, it will be the case that $f(x) < g(x)$ (see Figure 2(b)).

When thinking in terms of the vertical direction, “greater than” is equivalent to saying “above.”

When thinking in terms of the vertical direction, “less than” is equivalent to saying “below.”

---

14 When thinking in terms of the vertical direction, “greater than” is equivalent to saying “above.”

15 When thinking in terms of the vertical direction, “less than” is equivalent to saying “below.”

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Finally, if we select the \( x \)-value of the point of intersection of the graphs of \( f \) and \( g \), then for this value of \( x \), it is the case that \( f(x) \) and \( g(x) \) are equal; that is, \( f(x) = g(x) \) (see Figure 3).

![Figure 3](image)

**Figure 3.** The function values \( f(x) \) and \( g(x) \) are equal where the graphs of \( f \) and \( g \) intersect.

Let’s summarize our findings.

**Summary 2.**

- The solution of the equation \( f(x) = g(x) \) is the set of all \( x \) for which the graphs of \( f \) and \( g \) intersect.
- The solution of the inequality \( f(x) < g(x) \) is the set of all \( x \) for which the graph of \( f \) lies below the graph of \( g \).
- The solution of the inequality \( f(x) > g(x) \) is the set of all \( x \) for which the graph of \( f \) lies above the graph of \( g \).

Let’s look at an example.

**Example 3.** Given the graphs of \( f \) and \( g \) in Figure 4(a), use both set-builder and interval notation to describe the solution of the inequality \( f(x) < g(x) \). Then find the solutions of the inequality \( f(x) > g(x) \) and the equation \( f(x) = g(x) \) in a similar fashion.

To find the solution of \( f(x) < g(x) \), we must locate where the graph of \( f \) lies below the graph of \( g \). We draw a dashed vertical line through the point of intersection of the graphs of \( f \) and \( g \) (see Figure 4(b)), then note that the graph of \( f \) lies below the graph of \( g \) to the left of this dashed line. Consequently, the solution of the inequality \( f(x) < g(x) \) is the collection of all \( x \) that lie to the left of the dashed line. This set is shaded in red (or in a thicker line style if viewing in black and white) on the \( x \)-axis in Figure 4(b).
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Figure 4. Comparing $f$ and $g$.

(a) The graphs of $f$ and $g$.  
(b) The solution of $f(x) < g(x)$.

Note that the shaded points on the $x$-axis have $x$-values less than 2. Hence, the solution of $f(x) < g(x)$ is

$$(-\infty, 2) = \{ x : x < 2 \}.$$  

In like manner, the solution of $f(x) > g(x)$ is found by noting where the graph of $f$ lies above the graph of $g$ and shading the corresponding $x$-values on the $x$-axis (see Figure 5(a)). The solution of $f(x) > g(x)$ is $(2, \infty)$, or alternatively, $\{ x : x > 2 \}$.

To find the solution of $f(x) = g(x)$, note where the graph of $f$ intersects the graph of $g$, then shade the $x$-value of this point of intersection on the $x$-axis (see Figure 5(b)). Therefore, the solution of $f(x) = g(x)$ is $\{ x : x = 2 \}$. This is not an interval, so it is not appropriate to describe this solution with interval notation.

Figure 5. Further comparisons.

Let’s look at another example.
Example 4. Given the graphs of $f$ and $g$ in Figure 6(a), use both set-builder and interval notation to describe the solution of the inequality $f(x) > g(x)$. Then find the solutions of the inequality $f(x) < g(x)$ and the equation $f(x) = g(x)$ in a similar fashion.

![Graphs of $f$ and $g$](image1)

(a) The graphs of $f$ and $g$

![Solution of $f(x) > g(x)$](image2)

(b) The solution of $f(x) > g(x)$.

**Figure 6.** Comparing $f$ and $g.$

To determine the solution of $f(x) > g(x)$, we must locate where the graph of $f$ lies above the graph of $g$. Draw dashed vertical lines through the points of intersection of the graphs of $f$ and $g$ (see Figure 6(b)), then note that the graph of $f$ lies above the graph of $g$ between the dashed vertical lines just drawn. Consequently, the solution of the inequality $f(x) > g(x)$ is the collection of all $x$ that lie between the dashed vertical lines. We have shaded this collection on the $x$-axis in red (or with a thicker line style for those viewing in black and white) in Figure 6(b).

Note that the points shaded on the $x$-axis in Figure 6(b) have $x$-values between $-2$ and $3$. Consequently, the solution of $f(x) > g(x)$ is

$$(-2, 3) = \{x : -2 < x < 3\}.$$

In like manner, the solution of $f(x) < g(x)$ is found by noting where the graph of $f$ lies below the graph of $g$ and shading the corresponding $x$-values on the $x$-axis (see Figure 7(a)). Thus, the solution of $f(x) < g(x)$ is

$$(-\infty, -2) \cup (3, \infty) = \{x : x < -2 \text{ or } x > 3\}.$$

To find the solution of $f(x) = g(x)$, note where the graph of $f$ intersects the graph of $g$, and shade the $x$-value of each point of intersection on the $x$-axis (see Figure 7(b)). Therefore, the solution of $f(x) = g(x)$ is $\{x : x = -2 \text{ or } x = 3\}$. Because this solution set is not an interval, it would be inappropriate to describe it with interval notation.
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Solving Equations and Inequalities with the Graphing Calculator

We now know that the solution of \( f(x) = g(x) \) is the set of all \( x \) for which the graphs of \( f \) and \( g \) intersect. Therefore, the graphing calculator becomes an indispensable tool when solving equations.

Example 5. Use a graphing calculator to solve the equation

\[
1.23x - 4.56 = 5.28 - 2.35x. \quad (6)
\]

Note that equation (6) has the form \( f(x) = g(x) \), where

\[
f(x) = 1.23x - 4.56 \quad \text{and} \quad g(x) = 5.28 - 2.35x.
\]

Thus, our approach will be to draw the graphs of \( f \) and \( g \), then find the \( x \)-value of the point of intersection.

First, load \( f(x) = 1.23x - 4.56 \) into Y1 and \( g(x) = 5.28 - 2.35x \) into Y2 in the Y= menu of your graphing calculator (see Figure 8(a)). Select 6:ZStandard in the ZOOM menu to produce the graphs in Figure 8(b).

Figure 7. Further comparisons.

Figure 8. Sketching the graphs of \( f(x) = 1.23x - 4.56 \) and \( g(x) = 5.28 - 2.35x \).
The solution of equation (6) is the \(x\)-value of the point of intersection of the graphs of \(f\) and \(g\) in Figure 8(b). We will use the \texttt{intersect} utility in the \texttt{CALC} menu on the graphing calculator to determine the coordinates of the point of intersection.

We proceed as follows:

- Select \texttt{2nd CALC} (push the \texttt{2nd} button, followed by the \texttt{TRACE} button), which opens the menu shown in Figure 9(a).
- Select \texttt{5:intersect}. The calculator responds by placing the cursor on one of the graphs, then asks if you want to use the selected curve. You respond in the affirmative by pressing the \texttt{ENTER} key on the calculator.
- The calculator responds by placing the cursor on the second graph, then asks if you want to use the selected curve. Respond in the affirmative by pressing the \texttt{ENTER} key.
- The calculator responds by asking you to make a guess. In this case, there are only two graphs on the calculator, so any guess is appropriate.\(^{16}\) Simply press the \texttt{ENTER} key to use the current position of the cursor as your guess.

![Figure 9. Using the \texttt{intersect} utility.](image)

The result of this sequence of steps is shown in Figure 10. The coordinates of the point of intersection are approximately \((2.7486034, -1.179218)\). The \(x\)-value of this point of intersection is the solution of equation (6). That is, the solution of \(1.23x - 4.56 = 5.28 - 2.35x\) is approximately \(x \approx 2.7486034.\(^{17}\)

![Figure 10. The coordinates of the point of intersection.](image)

\(^{16}\) We will see in the case where there are two points of intersection, that the guess becomes more important.

\(^{17}\) It is important to remember that every time you pick up your calculator, you are only approximating a solution.

\(^{18}\) Please use a ruler to draw all lines.
Summary 7. Guidelines. You’ll need to discuss expectations with your teacher, but we expect our students to summarize their results as follows.

1. Set up a coordinate system. Label and scale each axis with xmin, xmax, ymin, and ymax.
2. Copy the image in your viewing window onto your coordinate system. Label each graph with its equation.
3. Draw a dashed vertical line through the point of intersection.
4. Shade and label the solution of the equation on the x-axis.

The result of following this standard is shown in Figure 11.

![Graph](attachment:image.png)

Figure 11. Summarizing the solution of equation (6).

Let’s look at another example.

Example 8. Use set-builder and interval notation to describe the solution of the inequality

\[0.85x^2 - 3 \geq 1.23x + 1.25. \tag{9}\]

Note that the inequality (9) has the form \(f(x) \geq g(x)\), where

\[f(x) = 0.85x^2 - 3 \quad \text{and} \quad g(x) = 1.23x + 1.25.\]

Load \(f(x) = 0.85x^2 - 3\) and \(g(x) = 1.23x + 1.25\) into Y1 and Y2 in the Y= menu, respectively, as shown in Figure 12(a). Select 6:ZStandard from the ZOOM menu to produce the graphs shown in Figure 12(b).

To find the points of intersection of the graphs of \(f\) and \(g\), we follow the same sequence of steps as we did in Example 5 up to the point where the calculator asks you to make a guess (i.e., 2nd CALC, 5:intersect, First curve ENTER, Second curve ENTER). Because there are two points of intersection, when the calculator asks you to
make a guess, you must move your cursor (with the arrow keys) so that it is closer to the point of intersection you wish to find than it is to the other point of intersection. Using this technique produces the two points of intersection found in Figures 13(a) and (b).

The approximate coordinates of the first point of intersection are \((-1.626682, -0.7508192)\). The second point of intersection has approximate coordinates \((3.0737411, 5.0307015)\).

It is important to remember that every time you pick up your calculator, you are only getting an approximation. It is possible that you will get a slightly different result for the points of intersection. For example, you might get \((-1.626685, -0.7508187)\) for your point of intersection. Based on the position of the cursor when you marked the curves and made your guess, you can get slightly different approximations. Note that this second solution is very nearly the same as the one we found, differing only in the last few decimal places, and is perfectly acceptable as an answer.

We now summarize our results by creating a coordinate system, labeling the axes, and scaling the axes with the values of the window parameters \(x_{\text{min}}, x_{\text{max}}, y_{\text{min}},\) and \(y_{\text{max}}\). We copy the image in our viewing window onto this coordinate system, labeling each graph with its equation. We then draw dashed vertical lines through each point of intersection, as shown in Figure 14.

We are solving the inequality \(0.85x^2 - 3 \geq 1.23x + 1.25\). The solution will be the union of the solutions of \(0.85x^2 - 3 > 1.23x + 1.25\) and \(0.85x^2 - 3 = 1.23x + 1.25\).

- To solve \(0.85x^2 - 3 > 1.23x + 1.25\), we note where the graph of \(y = 0.85x^2 - 3\) lies above the graph of \(y = 1.23x + 1.25\) and shade the corresponding \(x\)-values.
on the $x$-axis. In this case, the graph of $y = 0.85x^2 - 3$ lies above the graph of $y = 1.23x + 1.25$ for values of $x$ that lie outside of our dashed vertical lines.

- To solve $0.85x^2 - 3 = 1.23x + 1.25$, we note where the graph of $y = 0.85x^2 - 3$ intersects the graph of $y = 1.23x + 1.25$ and shade the corresponding $x$-values on the $x$-axis. This is why the points at $x \approx -1.626682$ and $x \approx 3.0737411$ are “filled.”

Thus, all values of $x$ that are either less than or equal to $-1.626682$ or greater than or equal to $3.0737411$ are solutions. That is, the solution of inequality $0.85x^2 - 3 > 1.23x + 1.25$ is approximately

$(-\infty, -1.626682] \cup [3.0737411, \infty) = \{ x : x \leq -1.626682 \text{ or } x \geq 3.0737411 \}$.

**Comparing Functions with Zero**

When we evaluate a function $f$ at a particular value of $x$, only one of three outcomes is possible. Either

$$f(x) = 0, \text{ or } f(x) > 0, \text{ or } f(x) < 0.$$ 

That is, either $f(x)$ equals zero, or $f(x)$ is positive, or $f(x)$ is negative. There are no other possibilities.

We could start fresh, taking a completely new approach, or we can build on what we already know. We choose the latter approach. Suppose that we are asked to compare $f(x)$ with zero? Is it equal to zero, is it greater than zero, or is it smaller than zero?

We set $g(x) = 0$. Now, if we want to compare the function $f$ with zero, we need only compare $f$ with $g$, which we already know how to do. To find where $f(x) = g(x)$, we note where the graphs of $f$ and $g$ intersect, to find where $f(x) > g(x)$, we note where the graph of $f$ lies above the graph of $g$, and finally, to find where $f(x) < g(x)$, we simply note where the graph of $f$ lies below the graph of $g$. 
However, the graph of \( g(x) = 0 \) is a horizontal line coincident with the \( x \)-axis. Indeed, \( g(x) = 0 \) is the equation of the \( x \)-axis. This argument leads to the following key results.

**Summary 10.**

- The solution of \( f(x) = 0 \) is the set of all \( x \) for which the graph of \( f \) intersects the \( x \)-axis.
- The solution of \( f(x) > 0 \) is the set of all \( x \) for which the graph of \( f \) lies strictly above the \( x \)-axis.
- The solution of \( f(x) < 0 \) is the set of all \( x \) for which the graph of \( f \) lies strictly below the \( x \)-axis.

For example:

- To find the solution of \( f(x) = 0 \) in **Figure 15**(a), we simply note where the graph of \( f \) crosses the \( x \)-axis in **Figure 15**(a). Thus, the solution of \( f(x) = 0 \) is \( x = 1 \).
- To find the solution of \( f(x) > 0 \) in **Figure 15**(b), we simply note where the graph of \( f \) lies above the \( x \)-axis in **Figure 15**(b), which is to the right of the vertical dashed line through \( x = 1 \). Thus, the solution of \( f(x) > 0 \) is \( (1, \infty) = \{ x : x > 1 \} \).
- To find the solution of \( f(x) < 0 \) in **Figure 15**(c), we simply note where the graph of \( f \) lies below the \( x \)-axis in **Figure 15**(c), which is to the left of the vertical dashed line at \( x = 1 \). Thus, the solution of \( f(x) < 0 \) is \( (-\infty, 1) = \{ x : x < 1 \} \).

![Figure 15](image_url)

**Figure 15.** Comparing the function \( f \) with zero.

We next define some important terminology.

**Definition 11.** If \( f(a) = 0 \), then \( a \) is called a zero of the function \( f \). The graph of \( f \) will intercept the \( x \)-axis at \((a, 0)\), a point called the x-intercept of the graph of \( f \).

Your calculator has a utility that will help you to find the zeros of a function.
Example 12. Use a graphing calculator to solve the inequality

\[ 0.25x^2 - 1.24x - 3.84 \leq 0.\]

Note that this inequality has the form \( f(x) \leq 0 \), where \( f(x) = 0.25x^2 - 1.24x - 3.84 \). Our strategy will be to draw the graph of \( f \), then determine where the graph of \( f \) lies below or on the \( x \)-axis.

We proceed as follows:

- First, load the function \( f(x) = 0.25x^2 - 1.24x - 3.84 \) into the \( Y_1 \) in the \( Y= \) menu of your calculator. Select \( 6:Z\text{Standard} \) from the \( Z\text{OOM} \) menu to produce the image in Figure 16(a).
- Press \( 2\text{nd} \) \( \text{CALC} \) to open the menu shown in Figure 16(b), then select \( 2:\text{zero} \) to start the utility that will find a zero of the function (an \( x \)-intercept of the graph).
- The calculator asks for a “Left Bound,” so use your arrow keys to move the cursor slightly to the left of the leftmost \( x \)-intercept of the graph, as shown in Figure 16(c). Press \( \text{ENTER} \) to record this “Left Bound.”
- The calculator then asks for a “Right Bound,” so use your arrow keys to move the cursor slightly to the right of the \( x \)-intercept, as shown in Figure 16(d). Press \( \text{ENTER} \) to record this “Right Bound.”

![Figure 16](a) (b) (c) (d)

Figure 16. Finding a zero or \( x \)-intercept with the calculator.

- The calculator responds by marking the left and right bounds on the screen, as shown in Figure 17(a), then asks you to make a reasonable starting guess for the zero or \( x \)-intercept. You may use the arrow keys to move your cursor to any point, so long as the cursor remains between the left- and right-bound marks on the viewing window. We usually just leave the cursor where it is and press the \( \text{ENTER} \) to record this guess. We suggest you do that as well.
- The calculator responds by finding the coordinates of the \( x \)-intercept, as shown in Figure 17(b). Note that the \( x \)-coordinate of the \( x \)-intercept is approximately \(-2.157931\).
- Repeat the procedure to find the coordinates of the rightmost \( x \)-intercept. The result is shown in Figure 17(c). Note that the \( x \)-coordinate of the intercept is approximately \( 7.1179306 \).

The final step is the interpretation of results and recording of our solution on our homework paper. Referring to the Summary 7 Guidelines, we come up with the graph shown in Figure 18.
Several comments are in order. Noting that \( f(x) = 0.25x^2 - 1.24x - 3.84 \), we note:

1. The solutions of \( f(x) = 0 \) are the points where the graph crosses the \( x \)-axis. That’s why the points \((-2.157931, 0)\) and \((7.1179306, 0)\) are shaded and filled in Figure 18.

2. The solutions of \( f(x) < 0 \) are those values of \( x \) for which the graph of \( f \) falls strictly below the \( x \)-axis. This occurs for all values of \( x \) between \(-2.157931\) and \(7.1179306\). These points are also shaded on the \( x \)-axis in Figure 18.

3. Finally, the solution of \( f(x) \leq 0 \) is the union of these two shadings, which we describe in interval and set-builder notation as follows:

\[
[-2.157931, 7.1179306] = \{ x : -2.157931 \leq x \leq 7.1179306 \}
\]
2.4 Exercises

In Exercises 1-6, you are given the definition of two functions \( f \) and \( g \). Compare the functions, as in Example 1 of the narrative, at the given values of \( x \).

1. \( f(x) = x + 2 \), \( g(x) = 4 - x \) at \( x = -3, 1, \) and 2.

2. \( f(x) = 2x - 3 \), \( g(x) = 3 - x \) at \( x = -4, 2, \) and 5.

3. \( f(x) = 3 - x \), \( g(x) = x + 9 \) at \( x = -4, -3, \) and -2.

4. \( f(x) = x^2 \), \( g(x) = 4x + 5 \) at \( x = -2, 1, \) and 6.

5. \( f(x) = x^2 \), \( g(x) = -3x - 2 \) at \( x = -3, -1, \) and 0.

6. \( f(x) = |x| \), \( g(x) = 4 - x \) at \( x = 1, 2, \) and 3.

In Exercises 7-12, perform each of the following tasks. Remember to use a ruler to draw all lines.

i. Make an accurate copy of the image on graph paper (label each equation, label and scale each axis), drop a dashed vertical line through the point of intersection, then label and shade the solution of \( f(x) < g(x) \) on the \( x \)-axis. Use set-builder and interval notation to describe your solution set.

ii. Make a second copy of the image on graph paper, drop a dashed, vertical line through the point of intersection, then label and shade the solution of \( f(x) > g(x) \) on the \( x \)-axis. Use set-builder and interval notation to describe your solution set.

iii. Make a third copy of the image on graph paper, drop a dashed, vertical line through the point of intersection, then label and shade the solution of \( f(x) = g(x) \) on the \( x \)-axis.
In Exercises 13-16, perform each of the following tasks. *Remember to use a ruler to draw all lines.*

i. Make an accurate copy of the image on graph paper, drop dashed, vertical lines through the points of intersection, then label and shade the solution of $f(x) \geq g(x)$ on the $x$-axis. Use set-builder and interval notation to describe your solution set.

ii. Make a second copy of the image on graph paper, drop dashed, vertical lines through the points of intersection, then label and shade the solution of $f(x) < g(x)$ on the $x$-axis. Use set-builder and interval notation to describe your solution set.
13.\[\begin{align*}
&x = 5 \\
&y = 5
\end{align*}\]

14.\[\begin{align*}
&x = 5 \\
&y = 5
\end{align*}\]

15.\[\begin{align*}
&x = 5 \\
&y = 5
\end{align*}\]

16.\[\begin{align*}
&x = 5 \\
&y = 5
\end{align*}\]

In Exercises 17-20, perform each of the following tasks. Remember to use a ruler to draw all lines.

i. Load each side of the equation into the $Y=$ menu of your calculator. Adjust the WINDOW parameters so that the point of intersection of the graphs is visible in the viewing window. Use the \texttt{intersect} utility in the \texttt{CALC} menu of your calculator to determine the $x$-coordinate of the point of intersection.

ii. Make an accurate copy of the image in your viewing window on your homework paper. Label and scale each axis with $x_{\text{min}}, x_{\text{max}}, y_{\text{min}},$ and $y_{\text{max}},$ and label each graph with its equation.

iii. Draw a dashed, vertical line through the point of intersection. Shade and label the solution of the equation on the $x$-axis.

17. \[1.23x - 4.56 = 3.46 - 2.3x\]

18. \[2.23x - 1.56 = 5.46 - 3.3x\]

19. \[5.46 - 1.3x = 2.2x - 5.66\]

20. \[2.46 - 1.4x = 1.2x - 2.66\]
In Exercises 21-26, perform each of the following tasks. Remember to use a ruler to draw all lines.

i. Load each side of the inequality into the Y= menu of your calculator. Adjust the WINDOW parameters so that the point(s) of intersection of the graphs is visible in the viewing window. Use the intersect utility in the CALC menu of your calculator to determine the coordinates of the point(s) of intersection.

ii. Make an accurate copy of the image in your viewing window on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax, and label each graph with its equation.

iii. Draw a dashed, vertical line through the point(s) of intersection. Shade and label the solution of the inequality on the x-axis. Use both set-builder and interval notation to describe the solution set.

21. \(1.6x + 1.23 \geq -2.3x - 4.2\)

22. \(1.24x + 5.6 < 1.2 - 0.52x\)

23. \(0.15x - 0.23 > 8.2 - 0.6x\)

24. \(-1.23x - 9.76 \leq 1.44x + 22.8\)

25. \(0.5x^2 - 5 < 1.23 - 0.75x\)

26. \(4 - 0.5x^2 \leq 0.72x - 1.34\)

In Exercises 27-30, perform each of the following tasks. Remember to use a ruler to draw all lines.

i. Make an accurate copy of the image on graph paper (label the graph with the letter \(f\) and label and scale each axis), drop a dashed vertical line through the \(x\)-intercept of the graph of \(f\), then label and shade the solution of \(f(x) = 0\) on the \(x\)-axis. Use set-builder notation to describe your solution.

ii. Make a second copy of the image on graph paper, drop a dashed, vertical line through the \(x\)-intercept of the graph of \(f\), then label and shade the solution of \(f(x) > 0\) on the \(x\)-axis. Use set-builder and interval notation to describe your solution set.

iii. Make a third copy of the image on graph paper, drop a dashed, vertical line through the \(x\)-intercept of the graph of \(f\), then label and shade the solution of \(f(x) < 0\) on the \(x\)-axis. Use set-builder and interval notation to describe your solution set.

27.
In Exercises 31-34, perform each of the following tasks. Remember to use a ruler to draw all lines.

i. Make an accurate copy of the image on graph paper, drop dashed, vertical lines through the $x$-intercepts, then label and shade the solution of $f(x) \geq 0$ on the $x$-axis. Use set-builder and interval notation to describe your solution set.

ii. Make a second copy of the image on graph paper, drop dashed, vertical lines through the $x$-intercepts, then label and shade the solution of $f(x) < 0$ on the $x$-axis. Use set-builder and interval notation to describe your solution set.
In Exercises 35-38, perform each of the following tasks. Remember to use a ruler to draw all lines.

i. Load the given function $f$ into the Y= menu of your calculator. Adjust the WINDOW parameters so that the $x$-intercept(s) of the graph of $f$ is visible in the viewing window. Use the zero utility in the CALC menu of your calculator to determine the coordinates of the $x$-intercept(s) of the graph of $f$.

ii. Make an accurate copy of the image in your viewing window on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax, and label the graph with its equation.

iii. Draw a dashed, vertical line through the $x$-intercept(s). Shade and label the solution of the inequality $f(x) > 0$ on the $x$-axis. Use both set-builder and interval notation to describe the solution set.

35. $f(x) = -1.25x + 3.58$
36. $f(x) = 1.34x - 4.52$
37. $f(x) = 1.25x^2 + 4x - 5.9125$
38. $f(x) = -1.32x^2 - 3.96x + 5.9532$

In Exercises 39-42, perform each of the following tasks. Remember to use a ruler to draw all lines.

i. Load the given function $f$ into the Y= menu of your calculator. Adjust the WINDOW parameters so that the $x$-intercept(s) of the graph of $f$ is visible in the viewing window. Use the zero utility in the CALC menu of your calculator to determine the coordinates of the $x$-intercept(s) of the graph of $f$.

ii. Make an accurate copy of the image
in your viewing window on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax, and label the graph with its equation.

iii. Draw a dashed, vertical line through the x-intercept(s). Shade and label the solution of the inequality \( f(x) \leq 0 \) on the x-axis. Use both set-builder and interval notation to describe the solution set.

39. \( f(x) = -1.45x - 5.6 \)

40. \( f(x) = 1.35x + 8.6 \)

41. \( f(x) = -1.11x^2 - 5.9940x + 1.2432 \)

42. \( f(x) = 1.22x^2 - 6.3440x + 1.3176 \)
2.4 Answers

1. $f(-3) < g(-3)$, $f(1) = g(1)$, and $f(2) > g(2)$.

3. $f(-4) > g(-4)$, $f(-3) = g(-3)$, and $f(-2) < g(-2)$.

5. $f(-3) > g(-3)$, $f(-1) = g(-1)$, and $f(0) > g(0)$.

7. The solution of $f(x) = g(x)$ is $x = 3$.

The solution of $f(x) < g(x)$ is $(-\infty, 3) = \{x : x < 3\}$.

9. The solution of $f(x) = g(x)$ is $x = -2$.

The solution of $f(x) > g(x)$ is $(3, \infty) = \{x : x > 3\}$.
The solution of \( f(x) > g(x) \) is \((-\infty, -2) = \{x : x < -2\}\). The solution of \( f(x) > g(x) \) is \((3, \infty) = \{x : x > 3\}\).

The solution of \( f(x) < g(x) \) is \((-2, \infty) = \{x : x > -2\}\). The solution of \( f(x) < g(x) \) is \((-\infty, 3) = \{x : x < 3\}\).

11. The solution of \( f(x) = g(x) \) is \( x = 3 \).

13. The solution of \( f(x) \geq g(x) \) is \([-3, 3] = \{x : -3 \leq x \leq 3\}\).
The solution of \( f(x) < g(x) \) is 
\((-\infty, -3) \cup (3, \infty)\)

\[= \{ x : x < -3 \text{ or } x > 3 \}. \]

15. The solution of \( f(x) \geq g(x) \) is 
\((-\infty, -2] \cup [2, \infty)\)

\[= \{ x : x \leq -2 \text{ or } x \geq 2 \}. \]

17. \( x = 2.271955 \)

19. \( x = 3.177143 \)
21. \([-1.392308, \infty) = \{x : x \geq -1.392308\}\]

27. The solution of \(f(x) = 0\) is \(x = -1\).

23. \((11.24, \infty) = \{x : x > 11.24\}\)

The solution of \(f(x) > 0\) is \((-1, \infty) = \{x : x > -1\}\).

25. \((-4.358670, 2.858670) = \{x : -4.358670 < x < 2.858670\}\)

The solution of \(f(x) < 0\) is \((-\infty, -1) = \{x : x < -1\}\)
29. The solution of $f(x) = 0$ is $x = 2$.

The solution of $f(x) > 0$ is $(-\infty, 2) = \{x : x < 2\}$.

The solution of $f(x) < 0$ is $(2, \infty) = \{x : x > 2\}$.

31. The solution of $f(x) \geq 0$ is $[-3, 2] = \{x : -3 \leq x \leq 2\}$.

The solution of $f(x) < 0$ is $(-\infty, -3) \cup (2, \infty) = \{x : x < -3 \text{ or } x > 2\}$.

33. The solution of $f(x) \geq 0$ is $(-\infty, -2] \cup [1, \infty) = \{x : x \leq -2 \text{ or } x \geq 1\}$. 
The solution of \( f(x) < 0 \) is \((-2, 1) = \{x : -2 < x < 1\}\).

35. \((-\infty, 2.8640) = \{x : x < 2.8640\}\)

37. \((-\infty, -4.3) \cup (1.1, \infty) = \{x : x < -4.3 \text{ or } x > 1.1\}\)

39. \([-3.8621, \infty) = \{x : x \geq -3.8621\}\)

41. \((-\infty, -5.6] \cup [0.2, \infty) = \{x : x \leq -5.6 \text{ or } x \geq 0.2\}\)
2.5 Vertical Transformations

In this section we study the art of transformations: scalings, reflections, and translations. We will restrict our attention to transformations in the vertical or $y$-direction. Our goal is to apply certain transformations to the equation of a function, then ask what effect it has on the graph of the function.

We begin our task with an example that requires that we read the graph of a function to capture several key points that lie on the graph of the function.

**Example 1.** Consider the graph of $f$ presented in Figure 1(a). Use the graph of $f$ to complete the table in Figure 1(b).

![Figure 1](https://example.com/fig1.png)

(a) The graph of $f$.  
(b) The table.

**Figure 1.** Reading key values from the graph of $f$.

To compute $f(-1)$, we would locate $-1$ on the $x$-axis, draw a vertical arrow to the graph of $f$, then a horizontal arrow to the $y$-axis, as shown in Figure 2(a). The $y$-value of this final destination is the value of $f(-1)$. That is, $f(-1) = 2$. This allows us to complete one entry in the table, as shown in Figure 2(b). Continue in this manner to complete all of the entries in the table. The result is shown in Figure 2(c).
Chapter 2 Functions

Vertical Scaling

In the narrative that follows, we will have repeated need of the graph in Figure 2(a) and the table in Figure 2(c). They characterize the basic function that will be the starting point for the concepts of scaling, reflection, and translation that we develop in this section. Consequently, let’s place them side-by-side for emphasis in Figure 3.

We are now going to scale the graph of \( f \) in the vertical direction.

**Example 2.** If \( y = f(x) \) has the graph shown in Figure 3(a), sketch the graph of \( y = 2f(x) \).

What do we do when we meet a graph whose shape we are unsure of? The answer to this question is we plot some points that satisfy the equation in order to get an
idea of the shape of the graph. With that thought in mind, let’s evaluate the function $y = 2f(x)$ at $x = -2$.

The letter $f$ refers to the original function shown in Figure 3(a) and the table in Figure 3(b) contains the values of that function at the given values of $x$. Thus, in computing $y = 2f(-2)$, the first step is to look up the value of $f(-2)$ in the table in Figure 3(b). There we find that $f(-2) = 0$. Thus, we can write

$$y = 2f(-2) = 2(0) = 0.$$ 

In similar fashion, let’s evaluate the function $y = 2f(x)$ at $x = -1$. First, look up the value of $f(-1)$ in the table in Figure 3(b). There we find that $f(-1) = 2$. Thus, we can write

$$y = 2f(-1) = 2(2) = 4.$$ 

We finish by evaluating the function $y = 2f(x)$ at $x = 0, 1, 2$. Each time you need to evaluate the function $f$ at a number, take the result from the table or graph in Figure 3. What follows are the evaluations of $y = 2f(x)$ at $x = -2, -1, 0, 1, 2$.

- $y = 2f(-2) = 2(0) = 0$
- $y = 2f(-1) = 2(2) = 4$
- $y = 2f(0) = 2(0) = 0$
- $y = 2f(1) = 2(-2) = -4$
- $y = 2f(2) = 2(0) = 0$

We can arrange these results in a table shown in Figure 4(b), then plot them in the figure shown in Figure 4(a).

At this point, there are a number of comparisons you can make.

1. Compare the data in the tables in Figure 3(b) and Figure 4(b). Note that the $x$-values are identical. In both tables, $x = -2, -1, 0, 1, 2$. However, note
that each \( y \)-value in the table in Figure 4(b) is precisely double the corresponding \( y \)-value in the table in Figure 3(b).

2. Compare the graphs in Figure 3(a) and Figure 4(a). Note that the \( y \)-value of each point in the graph of \( y = 2f(x) \) in Figure 4(a) is precisely double the \( y \)-value of the corresponding point in Figure 3(a).

Note the result. The graph of \( y = 2f(x) \) has been stretched vertically (away from the \( x \)-axis), both positively and negatively, by a factor of 2.

Let’s look at another example.

\[ \textbf{Example 3.} \quad \text{If } y = f(x) \text{ has the graph shown in Figure 3(a), sketch the graph of } y = \frac{1}{2}f(x). \]

Let’s begin by evaluating the function \( y = \frac{1}{2}f(x) \) at \( x = -2 \). First, look up the value of \( f(-2) \) in the table in Figure 3(b). There we find that \( f(-2) = 0 \). Thus, we can write

\[ y = \frac{1}{2}f(-2) = \frac{1}{2}(0) = 0. \]

In similar fashion, let’s evaluate the function \( y = \frac{1}{2}f(x) \) at \( x = -1 \). First, look up the value of \( f(-1) \) in the table in Figure 3(b). There we find that \( f(-1) = 2 \). Thus, we can write

\[ y = \frac{1}{2}f(-1) = \frac{1}{2}(2) = 1. \]

Continuing in this manner, we can evaluate the function \( y = \frac{1}{2}f(x) \) at \( x = 0, 1, \) and 2.

\[ y = \frac{1}{2}f(0) = \frac{1}{2}(0) = 0 \]
\[ y = \frac{1}{2}f(1) = \frac{1}{2}(-2) = -1 \]
\[ y = \frac{1}{2}f(2) = \frac{1}{2}(0) = 0 \]

The results are recorded in the table in Figure 5(b). Rather than double each value of \( y \) as did the function \( y = 2f(x) \) in Example 2, this function \( y = \frac{1}{2}f(x) \) halves each value of \( y \). The graph of \( y = \frac{1}{2}f(x) \) and a table of key points on the graph are presented in Figures 5(a) and (b), respectively.

Again, there are a number of comparisons.

1. Compare the data in the tables in Figure 5(b) and Figure 3(b). Note that the \( x \)-values are identical. In both tables \( x = -2, -1, 0, 1, \) and 2. However, note that each \( y \)-value in the table in Figure 5(b) is precisely half the corresponding \( y \)-value in the table in Figure 3(b).

2. When you compare the graph of \( y = \frac{1}{2}f(x) \) in Figure 5(a) with the original graph of \( y = f(x) \) in Figure 3(a), note that each point on the graph of \( y = \frac{1}{2}f(x) \) has a \( y \)-value that is precisely half of the corresponding \( y \)-value on the original graph of \( y = f(x) \) in Figure 3(a).
Section 2.5 Vertical Transformations

Let’s summarize our findings.

A Visual Summary — Vertical Scaling. Consider the images in Figure 6.

- In Figure 6(a), we see pictured the graph of the original function $y = f(x)$.
- In Figure 6(b), note that each key point on the graph of $y = 2f(x)$ has a $y$-value that is precisely double the $y$-value of the corresponding point on the graph of $y = f(x)$ in Figure 6(a).
- In Figure 6(c), note that each key point on the graph of $y = (1/2)f(x)$ has a $y$-value that is precisely half the $y$-value of the corresponding point on the graph of $y = f(x)$ in Figure 6(a).
- Note that the $x$-value of each transformed point remains the same.

Figure 6. The graph of $y = 2f(x)$ stretches vertically (away from the $x$-axis) by a factor of 2. The graph of $y = (1/2)f(x)$ compresses vertically (toward the $x$-axis) by a factor of 2.
The visual summary in Figure 6 makes sketching the graphs of \( y = 2f(x) \) and \( y = (1/2)f(x) \) an easy task.

- Given the graph of \( y = f(x) \), to sketch the graph of \( y = 2f(x) \), simply take each point on the graph of \( y = f(x) \) and double its \( y \)-value, keeping the same \( x \)-value.
- Given the graph of \( y = f(x) \), to sketch the graph of \( y = (1/2)f(x) \), simply take each point on the graph of \( y = f(x) \) and halve its \( y \)-value, keeping the same \( x \)-value.

Follow the same procedures for other scaling factors. For example, in the case of \( y = 3f(x) \), take each point on the graph of \( y = f(x) \) and multiply its \( y \)-value by 3, keeping the same \( x \)-value. On the other hand, to draw the graph of \( y = (1/3)f(x) \), take each point on the graph of \( f \) and multiply its \( y \)-value by 1/3, keeping the same \( y \)-value.

In general, we can state the following.

**Summary 4.** Suppose we are given the graph of \( y = f(x) \).

- If \( a > 1 \), then the graph of \( y = af(x) \) is stretched vertically (away from the \( x \)-axis), both positively and negatively, by a factor of \( a \).
- If \( 0 < a < 1 \), then the graph of \( y = af(x) \) is compressed vertically (toward the \( x \)-axis), both positively and negatively, by a factor of \( 1/a \).

The second item in Summary 4 warrants a word of explanation. Compare the general form \( y = af(x) \) with the function of Example 3, \( y = (1/2)f(x) \). In this case, \( a = 1/2 \), so

\[
\frac{1}{a} = \frac{1}{1/2} = 1 \times 2 = 2.
\]

The second item says that when \( 0 < a < 1 \), the graph of \( y = af(x) \) is compressed vertically by a factor of \( 1/a \). Indeed, this is exactly what happens in the case of \( y = (1/2)f(x) \), which is compressed by a factor of \( 1/(1/2) \), or 2.

**Vertical Reflections**

For convenience, we begin by repeating the original graph of \( y = f(x) \) and its accompanying data.

We are now going to reflect the graph in the vertical direction (across the \( x \)-axis).
Example 5. If \( y = f(x) \) has the graph shown in Figure 7(a), sketch the graph of \( y = -f(x) \).

To set up a table of points in preparation for the plot of \( y = -f(x) \), we’ll use exactly the same values of \( x \) that you see in the table in Figure 7(b), namely \( x = -2, -1, 0, 1, \) and \( 2 \).

To evaluate \( y = -f(x) \) at the first value of \( x \), namely \( x = -2 \), we make the following calculation,

\[
y = -f(-2) = -(0) = 0,
\]

where we’ve used the fact that \( f(-2) = 0 \) from the table in Figure 7(b). In similar fashion, we evaluate \( y = -f(x) \) at each of the remaining values of \( x \), namely \( x = -1, 0, 1, \) and \( 2 \).

\[
\begin{align*}
y &= -f(-1) = -(2) = -2 \\
y &= -f(0) = -(0) = 0 \\
y &= -f(1) = -(-2) = 2 \\
y &= -f(2) = -(0) = 0
\end{align*}
\]

We assemble these points in the table in Figure 8(b) and plot them in Figure 8(a).

Note that the graph of \( y = -f(x) \) in Figure 8(a) is a reflection of the graph of \( y = f(x) \) in Figure 7(a) across the \( x \)-axis.\(^{21}\)

\[^{21}\text{Be sure to note that this is a reflection of the graph of } y = f(x) \text{ across the } x \text{-axis. Note that a reflection of the graph of } y = f(x) \text{ across the } y \text{-axis gives the same result, but that’s not what we’ve done here. We’ll address reflections across the } y \text{-axis in the next section.}\]
Let’s summarize what we’ve learned about vertical reflections.

**A Visual Summary — Vertical Reflections.** Consider the images in Figure 9.

- In Figure 9(a), we see pictured the original graph of \( y = f(x) \).
- In Figure 9(b), the graph of \( y = -f(x) \) is a reflection of the graph of \( y = f(x) \) across the \( x \)-axis.

Thus, given the graph of \( y = f(x) \), it is a simple task to draw the graph of \( y = -f(x) \).

- To draw the graph of \( y = -f(x) \), take each point on the graph of \( y = f(x) \) and reflect it across the \( x \)-axis, keeping the \( x \)-value the same, but negating the \( y \)-value.
Vertical Translations

Translations are perhaps the easiest transformation of all. A translation is a “shift” or a “slide.” Pretend, for a moment, that you’ve placed a transparent sheet of thin plastic over a sheet of graph paper. You’ve drawn a Cartesian coordinate system on your graph paper, but you’ve plotted your graph on the transparent sheet of plastic. Now, “shift” or “slide” the transparency over your graph paper in a constant direction without rotating the transparency. This is what we mean by a “translation.” In this section, we will focus strictly on vertical translations.

For convenience, we begin by repeating the original graph of $y = f(x)$ and its accompanying data in Figure 10(a) and (b), respectively. We will now translate this graph in the vertical direction.

![Graph of $f(x)$ and table of key points](image)

**Figure 10.** The original graph of $f$ and a table of key points on the graph of $f$.

**Example 6.** If $y = f(x)$ has the graph shown in Figure 10(a), sketch the graph of $y = f(x) + 1$.

We will evaluate $y = f(x) + 1$ at the same values shown in the table in Figure 10(b), namely $x = -2, -1, 0, 1,$ and $2$. To evaluate $y = f(x) + 1$ at the first value of $x$, namely $x = -2$, we make the following calculation

$$y = f(-2) + 1 = 0 + 1 = 1,$$

where we’ve used that fact that $f(-2) = 0$ from the table in Figure 10(b). In similar fashion, we can evaluate $y = f(x) + 1$ at each of the remaining values of $x$, namely $x = -1, 0, 1,$ and $2$.

$$y = f(-1) + 1 = 2 + 1 = 3$$
$$y = f(0) + 1 = 0 + 1 = 1$$
$$y = f(1) + 1 = -2 + 1 = -1$$
$$y = f(2) + 1 = 0 + 1 = 1$$

We assemble these points in the table in Figure 11(b) and plot them in Figure 11(a).
When you compare the entries in the table in Figure 11(b) with the original values in the table in Figure 10(b), you’ll note that the $x$-values in each table are identical, but the $y$-values in the table in Figure 11(b) are all increased by 1. This makes sense, because these are the $y$-values of the points associated with the function $y = f(x) + 1$. Of course, all the $y$-values should be 1 larger than the $y$-values associated with the original equation $y = f(x)$.

Note the result. The graph of $y = f(x) + 1$ in Figure 11(a), when compared with the graph of $y = f(x)$ in Figure 10(a), is shifted 1 unit upwards.

Let’s look at another example.

▶ Example 7. If $y = f(x)$ has the graph shown in Figure 10(a), sketch the graph of $y = f(x) - 2$.

Evaluate the function $y = f(x) - 2$ at each value of $x$ in the table in Figure 10(b). At $x = -2$,

$$y = f(-2) - 2 = 0 - 2 = -2.$$  

In similar fashion, evaluate $y = f(x) - 2$ at each remaining $x$-value in the table in Figure 10(b).

$$y = f(-1) - 2 = 2 - 2 = 0$$  
$$y = f(0) - 2 = 0 - 2 = -2$$  
$$y = f(1) - 2 = -2 - 2 = -4$$  
$$y = f(2) - 2 = 0 - 2 = -2$$  

We assemble these points in the table in Figure 12(b) and plot them in Figure 12(a).

When you compare the entries in the table in Figure 12(b) with the original values in the table in Figure 10(b), you’ll note that the $x$-values in each table are identical, but the $y$-values in the table in Figure 12(b) are all decremented by 2. This makes
Section 2.5 Vertical Transformations

Figure 12. The graph of $y = f(x) - 2$ and a table of key points on the graph.

sense, because these are the $y$-values of the points associated with the function $y = f(x) - 2$. Of course, all the $y$-values should be 2 less than the $y$-values associated with the original equation $y = f(x)$.

Note the result. The graph of $y = f(x) - 2$ in Figure 12(a), when compared with the graph of $y = f(x)$ in Figure 10(a), is shifted downward 2 units.

Let’s summarize what we’ve learned about vertical translations.

A Visual Summary — Vertical Translations (Shifts). Consider the images in Figure 13.

- In Figure 13(a), we see pictured the graph of the original function $y = f(x)$.
- In Figure 13(b), note that each key point on the graph of $y = f(x) + 1$ has a $y$-value that is precisely 1 unit larger than the $y$-value of the corresponding point on the graph of $y = f(x)$ in in Figure 13(a).
- In Figure 13(c), note that each key point on the graph of $y = f(x) - 2$ has a $y$-value that is precisely 2 units smaller than the $y$-value of the corresponding point on the graph of $y = f(x)$ in in Figure 13(a).
- Note that the $x$-value of each transformed point remains the same.

The visual summary in Figure 13 makes sketching the graphs of $y = f(x) + 1$ and $y = f(x) - 2$ an easy task.

- Given the graph of $y = f(x)$, to sketch the graph of $y = f(x) + 1$, simply take each point on the graph of $y = f(x)$ and move it upwards 1 unit, keeping the same $x$-value.
- Given the graph of $y = f(x)$, to sketch the graph of $y = f(x) - 2$, simply take each point on the graph of $y = f(x)$ and move it downwards 2 units, keeping the same $x$-value.
Chapter 2 Functions

In general, we can state the following.

**Summary 8.** Suppose that we are given the graph of \( y = f(x) \) and suppose that \( c \) is any positive real number.

- The graph of \( y = f(x) + c \) is shifted \( c \) units upward from the graph of \( y = f(x) \).
- The graph of \( y = f(x) - c \) is shifted \( c \) units downward from the graph of \( y = f(x) \).

**Composing Transformations**

Sometimes we will want to perform one transformation, then take the result of the first transformation and apply a second transformation. Let’s look at an example.

**Example 9.** Consider the graph of \( y = f(x) \) presented in Figure 14.

![Figure 13. The graph of \( y = f(x) + 1 \) is formed by shifting (vertically) the graph of \( y = f(x) \) upward 1 unit. The graph of \( y = f(x) - 2 \) is formed by shifting (vertically) the graph of \( y = f(x) \) downward 2 units.](image)

In Figure 13, the graph of \( y = f(x) + 1 \) is formed by shifting (vertically) the graph of \( y = f(x) \) upward 1 unit. The graph of \( y = f(x) - 2 \) is formed by shifting (vertically) the graph of \( y = f(x) \) downward 2 units.

![Figure 14. The graph of \( y = f(x) \) that will be transformed in Example 9.](image)
Use the concepts discussed in the Visual Summaries to sketch the graph of \( y = -2f(x) \) without creating and referring to a table of points.

Note that the equation \( y = -2f(x) \) can be formed by a sequence of two transformations.

1. First, scale the original function \( y = f(x) \) to obtain the equation \( y = 2f(x) \).
2. Second, negate the resulting function \( y = 2f(x) \) to obtain the equation \( y = -2f(x) \).

Thus, the graph of \( y = -2f(x) \) can be formed as follows:

1. Start with the graph of \( y = f(x) \) and double the \( y \)-value of each point on the graph of \( y = f(x) \), keeping the same \( x \)-value. The result is the graph of \( y = 2f(x) \) shown in Figure 15(b).
2. Next, negate the \( y \)-value of each point on the graph of \( y = 2f(x) \), keeping the same \( x \)-value. The result is the graph of \( y = -2f(x) \) in Figure 15(c).

It is interesting to note that you will get the same result if you negate first, then scale the result. We will leave it to our readers to check that this is true.

Let’s look at one final example.

**Example 10.** Consider the graph of \( y = f(x) \) presented in Figure 16.

Use the concepts discussed in the Visual Summaries to sketch the graph of \( y = -f(x) + 2 \) without creating and referring to a table of points.

Note that the equation \( y = -f(x) + 2 \) can be formed by a sequence of two transformations.

1. First, negate the original function \( y = f(x) \) to obtain the equation \( y = -f(x) \).
2. Second, add 2 to the resulting function \( y = -f(x) \) to obtain the equation \( y = -f(x) + 2 \).

Thus, the graph of \( y = -f(x) + 2 \) can be formed as follows.
1. First, start with the graph of $y = f(x)$ in Figure 17(a) and negate the $y$-value of each point to produce the graph of $y = -f(x)$ Figure 17(b).
2. Next, add 2 to the $y$-value of each point on the graph of $y = -f(x)$ in Figure 17(b) to produce the graph of $y = -f(x) + 2$ in Figure 17(c).

Figure 17. Transforming the graph of $y = f(x)$, first reflecting across the $x$-axis, then shifting 2 units upward to obtain the graph of $y = -f(x) + 2$.

In Example 9, where we started with the graph of $y = f(x)$ and then graphed $y = 2f(x)$, the order of the transformations did not matter. Scale by 2, then negate, or negate and scale by 2, you get the same result (readers should verify this claim). However, in this example, the order in which the transformations are applied does matter. To see this, let’s do the following:

1. Add 2 to shift the graph of $y = f(x)$ in Figure 18(a) two units upward to obtain the graph of $y = f(x) + 2$ in Figure 18(b).
2. Negate the $y$-value of each point on the graph of $y = f(x) + 2$ in Figure 18(b) to obtain the graph of $y = -(f(x) + 2)$ in Figure 18(c). Note that we must negate the entire $y$-value. Hence the parentheses.
Unfortunately, the graph of \( y = -(f(x) + 2) \) in Figure 18(c) is not the same as the graph of \( y = -f(x) + 2 \) in Figure 17(c). But of course, this makes complete sense, as the equations (in the case of Figure 18(c))

\[
y = -(f(x) + 2) = -f(x) - 2
\]

and (in the case of Figure 17(c))

\[
y = -f(x) + 2 \tag{11}
\]

are also not the same.

![Graphs](image)

**Figure 18.** Transforming the graph of \( y = f(x) \), shifting 2 units upward to obtain the graph of \( y = f(x) + 2 \), then reflecting across the \( x \)-axis to obtain the graph of \( y = -(f(x) + 2) \).

Therefore, care must be taken when applying more than one transformation. Here is a good rule of thumb to live by.

**Do Vertical Scalings and Reflections First, then Vertical Translations.**

When performing a sequence of vertical transformations, it is usually easier (less confusing) to apply vertical scalings and reflections before vertical translations.

However, as long as you perform the transformations correctly, you should obtain the correct result. In Example 10, if you want to sketch the graph of \( y = -f(x) + 2 \) by doing the translation first, the correct way to proceed is as follows (though somewhat counterintuitive):

1. First, shift the graph of \( y = f(x) \) downward 2 units to obtain the graph of \( y = f(x) - 2 \).
2. Second, reflect the graph of \( y = f(x) - 2 \) across the \( x \)-axis to obtain the graph of \( y = -(f(x) - 2) \). Again, note the use of parentheses as we negate the entire \( y \)-value.

Finally, note that

\[
y = -(f(x) - 2) = -f(x) + 2.
\]
We will leave it to our readers to show that this sequence produces the correct result, a graph identical to the correct answer shown in Figure 17(c).

Summary

In this section we’ve seen how a handful of transformations greatly enhance our graphing capability. We end this section by listing the transformations presented in this section and their effects on the graph of a function.

**Vertical Transformations.** Suppose we are given the graph of $y = f(x)$.

- If $a > 1$, then the graph of $y = af(x)$ is stretched vertically (away from the $x$-axis), both positively and negatively, by a factor of $a$.
- If $0 < a < 1$, then the graph of $y = af(x)$ is compressed vertically (toward the $x$-axis), both positively and negatively, by a factor of $1/a$.
- The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ across the $x$-axis.
- If $c > 0$, then the graph of $y = f(x) + c$ is shifted $c$ units upward from the graph of $y = f(x)$.
- If $c > 0$, then the graph of $y = f(x) - c$ is shifted $c$ units downward from the graph of $y = f(x)$. 
2.5 Exercises

Pictured below is the graph of a function \( f \).

![Graph of function f]

The table that follows evaluates the function \( f \) in the plot at key values of \( x \). Notice the horizontal format, where the first point in the table is the ordered pair \((-4, 0)\).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-3</th>
<th>0</th>
<th>2</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>-4</td>
<td>-4</td>
<td>0</td>
</tr>
</tbody>
</table>

Use the graph and the table to complete each of following tasks for Exercises 1-10.

i. Set up a coordinate system on graph paper. Label and scale each axis, then copy and label the original graph of \( f \) onto your coordinate system. Remember to draw all lines with a ruler.

ii. Use the original table to help complete the table for the given function in the exercise.

iii. Using a different colored pencil, plot the data from your completed table on the same coordinate system as the original graph of \( f \). Use these points to help complete the graph of the given function in the exercise, then label this graph with its equation given in the exercise.

1. \( y = 2f(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-3</th>
<th>0</th>
<th>2</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
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</table>

2. \( y = \frac{1}{2}f(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-3</th>
<th>0</th>
<th>2</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
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</tbody>
</table>

3. \( y = -f(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-3</th>
<th>0</th>
<th>2</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
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</tbody>
</table>

4. \( y = f(x) - 2 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-3</th>
<th>0</th>
<th>2</th>
<th>5</th>
<th>6</th>
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<tbody>
<tr>
<td>( y )</td>
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</table>

5. \( y = f(x) + 4 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-3</th>
<th>0</th>
<th>2</th>
<th>5</th>
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<tr>
<td>( y )</td>
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6. \( y = -2f(x) \).

<table>
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<tr>
<th>( x )</th>
<th>-4</th>
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<th>0</th>
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<tr>
<td>( y )</td>
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</tbody>
</table>

22 Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/
7. \[ y = (-1/2)f(x). \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-3</th>
<th>0</th>
<th>2</th>
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<tbody>
<tr>
<td>( y )</td>
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</tbody>
</table>

8. \[ y = -f(x) + 3. \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-3</th>
<th>0</th>
<th>2</th>
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<tr>
<td>( y )</td>
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</tbody>
</table>

9. \[ y = -f(x) - 2. \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
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<th>0</th>
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<td>( y )</td>
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</tbody>
</table>

10. \[ y = (-1/2)f(x) + 3. \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-3</th>
<th>0</th>
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<tr>
<td>( y )</td>
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</table>

11. Use your graphing calculator to draw the graph of \( y = \sqrt{x} \). Then, draw the graph of \( y = -\sqrt{x} \). In your own words, explain what you learned from this exercise.

12. Use your graphing calculator to draw the graph of \( y = |x| \). Then, draw the graph of \( y = -|x| \). In your own words, explain what you learned from this exercise.

13. Use your graphing calculator to draw the graph of \( y = x^2 \). Then, in succession, draw the graphs of \( y = x^2 - 2 \), \( y = x^2 - 4 \), and \( y = x^2 - 6 \). In your own words, explain what you learned from this exercise.

14. Use your graphing calculator to draw the graph of \( y = x^2 \). Then, in succession, draw the graphs of \( y = x^2 + 2 \), \( y = x^2 + 4 \), and \( y = x^2 + 6 \). In your own words, explain what you learned from this exercise.

15. Use your graphing calculator to draw the graph of \( y = |x| \). Then, in succession, draw the graphs of \( y = 2|x| \), \( y = 3|x| \), and \( y = 4|x| \). In your own words, explain what you learned from this exercise.

16. Use your graphing calculator to draw the graph of \( y = |x| \). Then, in succession, draw the graphs of \( y = (1/2)|x| \), \( y = (1/3)|x| \), and \( y = (1/4)|x| \). In your own words, explain what you learned from this exercise.

Pictured below is the graph of a function \( f \). In Exercises 17-22, use this graph to perform each of the following tasks.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Make an exact copy of the graph of \( f \) on your coordinate system. Remember to draw all lines with a ruler.

ii. In the narrative, a shadow box at the end of the section summarizes the concepts and technique of vertical scaling, vertical reflection, and vertical translation. Use the shortcut ideas presented in this summary shadow box...
to draw the graphs of the functions that follow without using tables.

iii. Use a different colored pencil to draw the graph of the function given in the exercise. Label this graph with its equation. Be sure that key points are accurately plotted. In each exercise, please plot exactly two plots per coordinate system, the graph of original function \( f \) and the graph of the function in the exercise.

17. \( y = (1/2)f(x) \).
18. \( y = 2f(x) \).
19. \( y = -f(x) \).
20. \( y = f(x) - 1 \).
21. \( y = f(x) + 3 \).
22. \( y = f(x) - 4 \).

Pictured below is the graph of a function \( f \). In Exercises 23-28, use this graph to perform each of the following tasks.

23. \( y = 2f(x) \).
24. \( y = (1/2)f(x) \).
25. \( y = -f(x) \).
26. \( y = f(x) + 3 \).
27. \( y = f(x) - 2 \).
28. \( y = f(x) - 1 \).

Pictured below is the graph of a function \( f \). In Exercises 29-34, use this graph to perform each of the following tasks.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Make an exact copy of the graph of \( f \) on your coordinate system. Remember to draw all lines with a ruler.
ii. In the narrative, a shadow box at the end of the section summarizes the concepts and technique of vertical scaling, vertical reflection, and vertical translation. Use the shortcut ideas presented in this summary shadow box to draw the graphs of the functions that follow without using tables.

iii. Use a different colored pencil to draw the graph of the function given in the exercise. Label this graph with its equation. Be sure that key points are accurately plotted. In each exercise, please plot exactly two plots per coordinate system, the graph of original function \( f \) and the graph of the function in the exercise.
of graph paper. Label and scale each axis. Make an exact copy of the graph of $f$ on your coordinate system. Remember to draw all lines with a ruler.

ii. In the narrative, a shadow box at the end of the section summarizes the concepts and technique of vertical scaling, vertical reflection, and vertical translation. Use the shortcut ideas presented in this summary shadow box to draw the graphs of the functions that follow without using tables.

iii. Use a different colored pencil to draw the graph of the function given in the exercise. Label this graph with its equation. Be sure that key points are accurately plotted. In each exercise, please plot exactly two plots per coordinate system, the graph of original function $f$ and the graph of the function in the exercise.

29. $y = (-1/2)f(x)$.

30. $y = -2f(x)$.

31. $y = -f(x) + 2$.

32. $y = -f(x) - 3$.

33. $y = 2f(x) - 3$.

34. $y = (-1/2)f(x) + 1$. 
2.5 Answers

1. $y = 2f(x)$

3. $y = -f(x)$

5. $y = f(x) + 4$

7. $y = \frac{-1}{2}f(x)$
9. Multiplying by $-1$, as in $y = -\sqrt{x}$, reflects the graph across the $x$-axis.

11. Subtracting $c$, where $c > 0$, moves the graph $c$ units downward.

13. Multiply by a scalar $a$, such that $a$ is larger than 1, stretches the graph vertically by a factor of $a$. 

15. Multiply by a scalar $a$, such that $a$ is larger than 1, stretches the graph vertically by a factor of $a$. 

17. 

19. 

21. 

23. 

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25. \[ y = -f(x) \]

31. \[ y = f(x) + 2 \]

27. \[ y = f(x) - 2 \]

33. \[ y = 2f(x) - 3 \]

29. \[ y = -\frac{1}{2}f(x) \]
2.6 Horizontal Transformations

In the previous section, we introduced the concept of transformations. We made a change to the basic equation \( y = f(x) \), such as \( y = af(x) \), \( y = -f(x) \), \( y = f(x) - c \), or \( y = f(x) + c \), then studied how these changes affected the shape of the graph of \( y = f(x) \). In that section, we concentrated strictly on transformations that applied in the vertical direction. In this section, we will study transformations that will affect the shape of the graph in the horizontal direction.

We begin our task with an example that requires that we read the graph of a function to capture several key points that lie on the graph of the function.

Example 1. Consider the graph of \( f \) presented in Figure 1(a). Use the graph of \( f \) to complete the table in Figure 1(b).

![Figure 1](image)

(a) The graph of \( f \). (b) The table.

To compute \( f(-2) \), for example, we would first locate \(-2\) on the \( x \)-axis, draw a vertical arrow to the graph of \( f \), then a horizontal arrow to the \( y \)-axis, as shown in Figure 2(a). The \( y \)-value of this final destination is the value of \( f(-2) \). That is, \( f(-2) = -4 \). This allows us to complete one entry in the table, as shown in Figure 2(b). Continue in this manner to complete all of the entries in the table. The result is shown in Figure 2(c).
(a) The graph of $f$.

(b) Recording $f(-2) = -4$.

(c) Completed table.

Figure 2. Recording coordinates of points on the graph of $f$ in the tables.

**Horizontal Scaling**

In the narrative that follows, we will have repeated need of the graph in Figure 2(a) and the table in Figure 2(c). They characterize the basic function that will be the starting point for the concepts of scaling, reflection, and translation that we develop in this section. Consequently, let’s place them side-by-side for emphasis in Figure 3.

We are now going to scale the graph of $f$ in the horizontal direction.

**Example 2.** If $y = f(x)$ has the graph shown in Figure 3(a), sketch the graph of $y = f(2x)$.

In the previous section, we investigated the graph of $y = 2f(x)$. The number 2 was outside the function notation and as a result we stretched the graph of $y = f(x)$ vertically by a factor of 2. However, note that the 2 is now inside the function notation.
$y = f(2x)$. Intuition would demand that this might have something to do with scaling in the $x$-direction (horizontal direction), but how?

Again, when we’re unsure of the shape of the graph, we rely on plotting a table of points. We begin by picking these $x$-values: $x = -2, -1, 0, 1, \text{ and } 2$. Note that these are precisely half of each of the $x$-values presented in the table in Figure 3(b). We will now evaluate the function $y = f(2x)$ at each of these $x$-values. For example, to compute $y = f(2x)$ at $x = -2$, we first insert $x = -2$ for $x$ to obtain

$$ y = f(2(-2)) = f(-4). $$

To complete the computation, we must now evaluate $f(-4)$. However, this result is recorded in the table in Figure 3(b). There we find that $f(-4) = 0$, and we can complete the computation started above.

$$ y = f(2(-2)) = f(-4) = 0 $$

In similar fashion, to evaluate the function $y = f(2x)$ at $x = -1$, first substitute $x = -1$ in $y = f(2x)$ to obtain

$$ y = f(2(-1)) = f(-2). $$

Now, note that $f(-2)$ is the next recorded value in the table in Figure 3(b). There we find that $f(-2) = -4$, so we can complete the computation started above.

$$ y = f(2(-1)) = f(-2) = -4 $$

At this point, you might see why we chose $x$-values: $-2, -1, 0, 1, \text{ and } 2$. These are precisely half of the $x$-values in the table of original values for the function $y = f(x)$ in Figure 3(b). When the values $-2, -1, 0, 1, \text{ and } 2$ are substituted into the function $y = f(2x)$, they are first doubled before we go to look up the function value in the table in Figure 3(b).

Continuing in this manner, we evaluate the function $y = f(2x)$ at the remaining values of $x$, namely, $0, 1, \text{ and } 2$.

$$ y = f(2(0)) = f(0) = 0, $$
$$ y = f(2(1)) = f(2) = 2, $$
$$ y = f(2(2)) = f(4) = 0 $$

We enter these values into the table in Figure 4(b) and plot them to determine the graph of $y = f(2x)$ in Figure 4(a).

At this point, there are a number of comparisons you can make.

1. Compare the data in the table in Figure 4(b) with the original function data in the table in Figure 3(b). Note that the $y$-values in each table are identical. However, note that each $x$-value in the table of Figure 4(b) is precisely half of the corresponding $x$-value in the table of Figure 3(b).

2. Compare the graph of $y = f(2x)$ in Figure 4(a) with the original graph of $y = f(x)$ in Figure 3(a). Note that each $x$-value at each point on the graph of $y = f(2x)$ in
Figure 4. The points in the table
are points on the graph of \( y = f(2x) \).

Figure 4(a) is precisely half the \( x \)-value of the corresponding point on the graph of \( y = f(x) \) in Figure 3(a).

Note the result. The graph of \( y = f(2x) \) is compressed horizontally (toward the \( y \)-axis), both positively and negatively, by a factor of 2. Note that this is exactly the opposite of what you might expect by intuition, but a careful examination of the data in the tables in Figures 3(b) and 4(b) will explain why.

Let’s look at another example.

Example 3. If \( y = f(x) \) has the graph shown in Figure 3(a), sketch the graph of \( y = f((1/2)x) \).

Rather than doubling each value of \( x \) at the start, this function first halves each value of \( x \). Thus, we will want to evaluate the function \( y = f((1/2)x) \) at \( x = -8, -4, 0, 4 \), and 8. For example, to evaluate the function \( y = f((1/2)x) \) at \( x = -8 \), first substitute \( x = -8 \) to obtain

\[
y = f((1/2)(-8)) = f(-4).
\]

Now, look up this value in the table in Figure 3(b) and note that \( f(-4) = 0 \). Thus, we can complete the computation as follows.

\[
y = f((1/2)(-8)) = f(-4) = 0
\]

Similarly, to evaluate the function \( y = f((1/2)x) \) at \( x = -4 \), first substitute \( x = -4 \) to obtain

\[
y = f((1/2)(-4)) = f(-2).
\]

Now, look up this value in the table in Figure 3(b) and note that \( f(-2) = -4 \). Thus, we can complete the computation as follows.
\[ y = f((1/2)(-4)) = f(-2) = -4 \]

At this point, you will see why we chose \( x \)-values: \(-8, -4, 0, 4, \) and \( 8 \). These values are precisely double the \( x \)-values in the table of original values for the function \( y = f(x) \) in Figure 3(b). When the values \(-8, -4, 0, 4, \) and \( 8 \) are substituted into the function \( y = f((1/2)x) \), they are first halved before we go to look up the function value in the table in Figure 3(b). This halving leads to the values \(-4, -2, 0, 2, \) and \( 4 \), which are precisely the values available in the table in Figure 3(b).

We make similar computations at the remaining values of \( x \), namely \( 0, 4, \) and \( 8 \).

\[
\begin{align*}
y &= f((1/2)(0)) = f(0) = 0 \\
y &= f((1/2)(4)) = f(2) = 2 \\
y &= f((1/2)(8)) = f(4) = 0
\end{align*}
\]

Hopefully, these computations explain our choice of \( x \)-values above. Each of these results is recorded in the table in Figure 5(b) and plotted on the graph shown in Figure 5(a).

\[
\begin{array}{|c|c|c|}
\hline
x & y = f((1/2)x) & (x, f((1/2)x)) \\
\hline
-8 & 0 & (-8, 0) \\
-4 & -4 & (-4, -4) \\
0 & 0 & (0, 0) \\
4 & 2 & (4, 2) \\
8 & 0 & (8, 0) \\
\hline
\end{array}
\]

Figure 5. The points in the table are points on the graph of \( y = f((1/2)x) \).

Again, note that the \( y \)-values in the table in Figure 5(b) are identical to the \( y \)-values in the table in Figure 3(b). However, each \( x \)-value in the table in Figure 5(b) is precisely double the corresponding \( x \)-value in the table in Figure 3(b).

This doubling of the \( x \)-values is apparent in the graph of \( y = f((1/2)x) \) shown in Figure 5(a), where the graph is stretched by a factor of 2 horizontally (away from the \( y \)-axis), both positively and negatively. Note that this is exactly the opposite of what you might expect by intuition, but a careful examination of the data in the tables in Figures 3(b) and 5(b) will explain why.

Let’s summarize our findings.
A Visual Summary — Horizontal Scaling. Consider the images in Figure 6.

- In Figure 6(a), we see pictured the graph of the original function \( y = f(x) \).
- In Figure 6(b), note that each key point on the graph of \( y = f(2x) \) has an \( x \)-value that is precisely half the \( x \)-value of the corresponding point on the graph of \( y = f(x) \) in Figure 6(a).
- In Figure 6(c), note that each key point on the graph of \( y = f\left(\frac{1}{2}x\right) \) has an \( x \)-value that is twice the \( x \)-value of the corresponding point on the graph of \( y = f(x) \) in Figure 6(a).
- Note that the \( y \)-value of each transformed point remains the same.

![Figure 6](image)

**Figure 6.** The graph of \( y = f(2x) \) compresses horizontally (toward the \( y \)-axis) by a factor of 2. The graph of \( y = f\left(\frac{1}{2}x\right) \) stretches horizontally (away from the \( y \)-axis) by a factor of 2.

The visual summary in Figure 6 makes sketching the graphs of \( y = f(2x) \) and \( y = f\left(\frac{1}{2}x\right) \) an easy task.

- Given the graph of \( y = f(x) \), to sketch the graph of \( y = f(2x) \), simply take each point on the graph of \( y = f(x) \) and cut its \( x \)-value in half, keeping the same \( y \)-value.
- Given the graph of \( y = f(x) \), to sketch the graph of \( y = f\left(\frac{1}{2}x\right) \), simply take each point on the graph of \( y = f(x) \) and double its \( x \)-value, keeping the same \( y \)-value.

Follow the same procedures for other scaling factors. For example, in the case of \( y = f(3x) \), take each point on the graph of \( y = f(x) \) and divide its \( x \)-value by 3, keeping the same \( y \)-value. On the other hand, to draw the graph of \( y = f\left(\frac{1}{3}x\right) \), take each point on the graph of \( f \) and multiply its \( x \)-value by 3, keeping the same \( y \)-value.

In general, we can state the following.
Summary 4. Suppose we are given the graph of \( y = f(x) \).

- If \( a > 1 \), the graph of \( y = f(ax) \) compresses horizontally (toward the \( y \)-axis), both positively and negatively, by a factor of \( a \).
- If \( 0 < a < 1 \), the graph of \( y = f(ax) \) stretches horizontally (away from the \( y \)-axis), both positively and negatively, by a factor of \( 1/a \).

In the case of the first item in Summary 4, when we compare the general form \( y = f(ax) \) with \( y = f(2x) \), we see that \( a = 2 \). In this case, note that \( a > 1 \) and the graph of \( y = f(2x) \) compresses horizontally by a factor of 2 when compared with the graph of \( y = f(x) \) (see Figure 6(b)).

In the case of the second item in Summary 4, when we compare the general form \( y = f(ax) \) with the equation \( y = f((1/2)x) \), we see that \( a = 1/2 \), so
\[
\frac{1}{a} = \frac{1}{1/2} = 2.
\]

The second item in Summary 4 says that when \( 0 < a < 1 \), the graph of \( y = f(ax) \) stretches horizontally by a factor of \( 1/a \). Indeed, this is exactly what happened in the case of \( y = f((1/2)x) \), which stretched in the horizontal direction by a factor of \( 1/(1/2) \), or 2 (see Figure 6(c)).

Horizontal Reflections

For convenience, we begin by repeating the original graph of \( y = f(x) \) and its accompanying data in Figure 7. We are now going to reflect the graph of \( y = f(x) \) in the horizontal direction (across the \( y \)-axis).

\[
\begin{array}{c|c|c}
\hline
x & f(x) & (x, f(x)) \\
\hline
-4 & 0 & (-4, 0) \\
-2 & -4 & (-2, -4) \\
0 & 0 & (0, 0) \\
2 & 2 & (2, 2) \\
4 & 0 & (4, 0) \\
\hline
\end{array}
\]

Figure 7. The original graph of \( f \) and a table of key points on the graph of \( f \).

Example 5. If \( y = f(x) \) has the graph shown in Figure 7(a), draw the graph of \( y = f(-x) \).
In the previous section, we were asked to draw the graph of \( y = -f(x) \). Note how the minus sign appears on the outside of the function. Clearly, the \( y \)-values of \( y = -f(x) \) must be opposite in sign to the \( y \)-values of \( y = f(x) \). That is why the graph of \( y = -f(x) \) was a reflection of the graph of \( y = f(x) \) across the \( x \)-axis.

However, in this example, the minus sign is inside the function, leaving one to intuit that it is the \( x \)-values, not the \( y \)-values, that are being negated. We will choose the following \( x \)-values: \( x = 4, 2, 0, -2, \) and \( -4 \). This is a bit deceptive, as it looks like we are choosing the same \( x \)-values, only in reverse order. This is not the case. We are choosing the negative of each \( x \)-value in the table in Figure 7(b).

To evaluate \( y = f(-x) \) at our first \( x \)-value, namely \( x = 4 \), we perform the following calculation. First substitute \( x = 4 \) to obtain
\[
y = f(-4) = f(-4).
\]
Now, look up this value in the table in Figure 7(b) and note that \( f(-4) = 0 \). Thus, we can complete the computation as follows.
\[
y = f(-4) = f(-4) = 0
\]
Similarly, to evaluate the function \( y = f(-x) \) at \( x = 2 \), first substitute \( x = 2 \) to obtain
\[
y = f(-2) = f(-2).
\]
Now, look up this value in the table in Figure 7(b) and note that \( f(-2) = -4 \). Thus, we can complete the computation as follows.
\[
y = f(-2) = f(-2) = -4
\]
At this point, you will see why we chose \( x \)-values: \( 4, 2, 0, -2, \) and \( -4 \). These values are the negatives of the \( x \)-values in the table of original values for the function \( y = f(x) \) in Figure 7(b). When the values \( 4, 2, 0, -2, \) and \( -4 \) are substituted into the function \( y = f(-x) \), they are first negated before we go to look up the function value in the table in Figure 7(b). This negating leads to the values \(-4, -2, 0, 2, \) and \( 4 \), which are precisely the values available in the table in Figure 7(b).

We make similar computations at the remaining values of \( x \), namely \( x = 0, -2, \) and \( -4 \).
\[
y = f(-0) = f(0) = 0
\]
\[
y = f(-(-2)) = f(2) = 2
\]
\[
y = f(-(-4)) = f(4) = 0
\]
We organize these points in the table in Figure 8(b), then plot them in Figure 8(a).

When you compare the entries in the table in Figure 8(b) with those in the table in Figure 7(b), note that the \( y \)-values appear in the same order, but the \( x \)-values of the table in Figure 7(b) have been negated in the table in Figure 8(b). This means that a former point such as \((-2, -4)\) is transformed to the point \((2, -4)\), which is a reflection of the point \((-2, -4)\) across the \( y \)-axis.
Section 2.6 Horizontal Transformations

Thus, to produce the graph of \( y = f(-x) \), simply reflect the graph of \( y = f(x) \) across the \( y \)-axis.

Let’s summarize what we’ve learned about horizontal reflections.

A Visual Summary — Horizontal Reflections. Consider the images in Figure 9.

- In Figure 9(a), we see pictured the original graph of \( y = f(x) \).
- In Figure 9(b), the graph of \( y = f(-x) \) is a reflection of the graph of \( y = f(x) \) across the \( y \)-axis.
Thus, given the graph of \( y = f(x) \), it is a simple task to draw the graph of \( y = f(-x) \).

- To draw the graph of \( y = f(-x) \), take each point on the graph of \( y = f(x) \) and reflect it across the \( y\)-axis, keeping the \( y\)-value the same, but negating the \( x\)-value.

**Horizontal Translations**

In the previous section, we saw that the graphs of \( y = f(x) + c \) and \( y = f(x) - c \) were vertical translations of the graph of \( y = f(x) \). If \( c \) is a positive number, then the graph of \( y = f(x) + c \) shifts \( c \) units upward while the graph of \( y = f(x) - c \) shifts \( c \) units downward.

In this section, we will study horizontal translations. For convenience, we begin by repeating the original graph of \( y = f(x) \) and its accompanying data in **Figure 10**.

![Figure 10. The original graph of \( f \) and a table of key points on the graph of \( f \)](image)

**Example 6.** If \( y = f(x) \) has the graph shown in **Figure 10(a)**, sketch the graph of \( y = f(x + 1) \).

In the previous section, we drew the graph of \( y = f(x) + 1 \). Note that in \( y = f(x) + 1 \), the number 1 is outside the function. The result was a graph that was shifted 1 unit upwards in the \( y\)-direction.

In this case, \( y = f(x + 1) \) and the 1 is inside the function notation, leading one to intuit that the translation might be in the horizontal direction (\( x\)-direction). But how?

Again, we will set up a table of points that satisfy the equation \( y = f(x + 1) \), then plot them. Because this function requires that we first add 1 to each \( x\)-value before inserting it into the function, we will choose \( x\)-values appropriately, namely \( x = -5, -3, -1, 1, \) and 3. In a moment, it will be clear why we have chosen these particular values of \( x \). Perhaps you already see why?

We need to evaluate the function \( y = f(x + 1) \) at each of these chosen values of \( x \). To evaluate \( y = f(x + 1) \) at the first value, namely \( x = -5 \), we insert \( x = -5 \) and make the calculation.
To complete the calculation, we must now evaluate \( f(-4) \). However, this result is recorded in the table in Figure 10(b). There we find that \( f(-4) = 0 \), and we can complete the calculation started above.

\[
y = f(-5 + 1) = f(-4) = 0
\]

In similar fashion, we can evaluate the function \( y = f(x+1) \) at \( x = -3 \). First, substitute \( x = -3 \) in \( y = f(x+1) \) to obtain

\[
y = f(-3 + 1) = f(-2).
\]

To complete the calculation, we must now evaluate \( f(-2) \). However, this result is recorded in the table in Figure 10(b). There we find that \( f(-2) = -4 \), and we can complete the calculation started above.

\[
y = f(-3 + 1) = f(-2) = -4
\]

At this point, you might see why we chose \( x \)-values: \(-5, -3, -1, 1, \) and \( 3 \). These are precisely one less than the \( x \)-values in the table of original values for the function \( y = f(x) \) in Figure 10(b). When the values \(-5, -3, -1, 1, \) and \( 3 \) are substituted into the function \( y = f(x+1) \), we first add 1 to each value before we go to look up the function value in the table in Figure 10(b). This adding of 1 leads to the values \(-4, -2, 0, 2, \) and \( 4 \), which are precisely the values available in the table in Figure 10(b).

Continuing in this manner, we evaluate the function \( y = f(x+1) \) at the remaining values of \( x \), namely, \(-1, 1, \) and \( 3 \).

\[
\begin{align*}
y &= f(-1 + 1) = f(0) = 0 \\
y &= f(1 + 1) = f(2) = 2 \\
y &= f(3 + 1) = f(4) = 0
\end{align*}
\]

We assemble these results in the table in Figure 11(b) and plot them in Figure 11(a).
When you compare the points on the graph of \( y = f(x+1) \) in the table in Figure 11(b) with the original points on the graph of \( y = f(x) \) in the table in Figure 10(b), note that the \( y \)-values are identical, but the \( x \)-values in the table in Figure 11(b) are all 1 unit less than the corresponding \( x \)-values in the table in Figure 10(b). It is no wonder that the graph of \( y = f(x+1) \) in Figure 11(a) is shifted 1 unit to the left of the original graph of \( y = f(x) \) in Figure 10(a).

Note that this is somewhat counterintuitive, because we're seemingly adding 1 to each \( x \)-value in \( y = f(x+1) \). Why doesn't the graph move one unit to the right? Well, a careful comparison of the \( x \)-values in the tables in Figures 10(b) and 11(b) reveals the answer. In order to use the data in the table in Figure 10(b), we must first subtract 1 from each \( x \)-value to produce the \( x \)-values in the table in Figure 11(b). This is why the graph of \( y = f(x+1) \) moves 1 unit to the left instead of 1 unit to the right.

You might also recall that the function \( y = f(2x) \) compressed by a factor of 2, which is also the opposite of what intuition might dictate. Similarly, the function \( y = f((1/2)x) \) stretches by a factor of 2, which also goes counter to intuition. With these thoughts in mind, it is not surprising that \( y = f(x+1) \) shifts one unit to the left.

Still, a comparison of the \( x \)-values in the tables in Figures 10(b) and 11(b) provide irrefutable evidence that the shift is 1 unit to the left.

Let’s look at another example.

**Example 7.** If \( y = f(x) \) has the graph shown in Figure 10(a), sketch the graph of \( y = f(x-2) \).

Again, we will set up a table of points that satisfy the equation \( y = f(x-2) \), then plot them. Because this function requires that we first subtract 2 from each \( x \)-value before inserting it into the function, we will choose \( x \)-values: \(-2, 0, 2, 4, \) and 6. We need to evaluate the function \( y = f(x-2) \) at each of these values of \( x \).

To evaluate \( y = f(x-2) \) at the first value, namely \( x = -2 \), insert \( x = -2 \) into the function \( y = f(x-2) \) to obtain

\[
y = f(-2 - 2) = f(-4).
\]

In the table in Figure 10(b), we find that \( f(-4) = 0 \), which allows us to complete the calculation above.

\[
y = f(-2 - 2) = f(-4) = 0
\]

In similar fashion, we evaluate \( y = f(x-2) \) at \( x = 0 \) to obtain

\[
y = f(0 - 2) = f(-2).
\]

In the table in Figure 10(b), we find that \( f(-2) = -4 \), which allows us to complete the calculation above.

\[
y = f(0 - 2) = f(-2) = -4
\]
Hopefully, you see why we chose the $x$-values: $-2, 0, 2, 4,$ and $6$. These values are 2 larger than the $x$-values in the table of original values for the function $y = f(x)$ in Figure 10(b). When the values $-2, 0, 2, 4,$ and $6$ are substituted into the function $y = f(x - 2)$, we first subtract 2 from each value before we go to look up the function value in the table in Figure 10(b). This subtracting of 2 leads to $-4, -2, 0, 2, \text{ and } 4$, precisely the values that are available in the table in Figure 10(b).

Continuing in this manner, we evaluate $y = f(x - 2)$ at the remaining values of $x$, namely, $x = 2, 4,$ and $6$.

\[
\begin{align*}
y &= f(2 - 2) = f(0) = 0 \\
y &= f(4 - 2) = f(2) = 2 \\
y &= f(6 - 2) = f(4) = 0
\end{align*}
\]

We assemble these results in the table in Figure 12(b) and plot them in Figure 12(a).

When you compare the points on the graph of $y = f(x - 2)$ in the table in Figure 12(b) with the original points on the graph of $y = f(x)$ in the table in Figure 10(b), note that the $y$-values are identical, but the $x$-values in the table in Figure 12(b) are all 2 larger than the corresponding $x$-values in the table in Figure 10(b). It is no wonder that the graph of $y = f(x - 2)$ in Figure 12(a) is shifted 2 units to the right of the original graph of $y = f(x)$ in Figure 10(a).

Again, this runs counterintuitive (why doesn’t the graph of $y = f(x - 2)$ shift 2 units to the left?), but a comparison of the $x$-values in the tables in Figures 10(b) and 12(b) clearly indicates a shift to the right.

Let’s summarize what we’ve learned about horizontal translations.
Visual Summary — Horizontal Translations (Shifts). Consider the images in Figure 13.

- In Figure 13(a), we see pictured the graph of the original function $y = f(x)$.
- In Figure 13(b), note that each point on the graph of $y = f(x + 1)$ has an $x$-value that is 1 unit less than the $x$-value of the corresponding point on the graph of $y = f(x)$ in Figure 13(a).
- In Figure 13(c), note that each point on the graph of $y = f(x - 2)$ has an $x$-value that is 2 units greater than the $x$-value of the corresponding point on the graph of $y = f(x)$ in Figure 13(a).
- Note that the $y$-value of each transformed point remains the same.

Figure 13. The graph of $y = f(x + 1)$ is formed by shifting (horizontally) the graph of $y = f(x)$ one unit to the left. The graph of $y = f(x - 2)$ is formed by shifting (horizontally) the graph of $y = f(x)$ two units to the right.

The visual summary in Figure 13 makes sketching the graphs of $y = f(x + 1)$ and $y = f(x - 2)$ an easy task.

- Given the graph of $y = f(x)$, to sketch the graph of $y = f(x + 1)$, simply take each point on the graph of $y = f(x)$ and shift it 1 unit to the left, keeping the same $y$-value.
- Given the graph of $y = f(x)$, to sketch the graph of $y = f(x - 2)$, simply take each point on the graph of $y = f(x)$ and shift it 2 units to the right, keeping the same $y$-value.

In general, we can state the following.

Summary 8. Suppose that we are given the graph of $y = f(x)$ and suppose that $c$ is any positive real number.

- The graph of $y = f(x + c)$ is shifted $c$ units to the left of the graph of $y = f(x)$.
- The graph of $y = f(x - c)$ is shifted $c$ units to the right of the graph of $y = f(x)$.
When we looked at vertical translations in the previous section, a translation was described by first imagining a graph on a sheet of transparent plastic, then sliding the transparency (without rotating it) over a coordinate system on a sheet of graph paper. Horizontal translations can be thought of in the same way, as sliding the graph on the transparency \( c \) units to the left, or \( c \) units to the right.

**Extra Practice**

In this section, let’s take the concepts from the Visual Summaries and put them to work on some final examples.

- **Example 9.** Consider the graph of \( f \) in Figure 14.

![Figure 14](image)

> **Figure 14.** The graph of \( y = f(x) \) for Example 9.

Use the concepts from the Visual Summaries (scaling, reflection, and translation) to sketch the graphs of \( y = f(2x) \), \( y = f(-x) \), and \( y = f(x + 2) \) without creating and referring to tables.

To sketch the graph of \( y = f(2x) \), simply take each point on the graph of \( y = f(x) \) in Figure 15(a) and divide its \( x \)-value by 2, keeping the same \( y \)-value. The result is shown in Figure 15(b).

![Figure 15](image)

> **Figure 15.** Compress the graph of \( y = f(x) \) by a factor of 2 to produce the graph of \( y = f(2x) \).
To sketch the graph of $y = f(-x)$, simply take each point on the graph of $y = f(x)$ in Figure 16(a) and negate its $x$-value, keeping the same $y$-value. The result is shown in Figure 16(b).

Figure 16. Reflect the graph of $y = f(x)$ across the $y$-axis to produce the graph of $y = f(-x)$.

To sketch the graph of $y = f(x + 2)$, simply take each point on the graph of $y = f(x)$ in Figure 17(a) and subtract 2 from its $x$-value, keeping the same $y$-value. The result is shown in Figure 17(b).

Figure 17. Shift the graph of $y = f(x)$ to the left 2 units to produce the graph of $y = f(x + 2)$.
Summary

In this section we’ve seen how a handful of transformations greatly enhance our graphing capability. We end this section by listing the transformations presented in this section and their effects on the graph of a function.

**Vertical Transformations.** Suppose we are given the graph of \( y = f(x) \).

- If \( a > 1 \), the graph of \( y = f(ax) \) compresses horizontally (toward the \( y \)-axis), both positively and negatively, by a factor of \( a \).
- If \( 0 < a < 1 \), the graph of \( y = f(ax) \) stretches horizontally (away from the \( y \)-axis), both positively and negatively, by a factor of \( 1/a \).
- The graph of \( y = f(-x) \) is a reflection of the graph of \( y = f(x) \) across the \( y \)-axis.
- If \( c > 0 \), then the graph of \( y = f(x + c) \) is shifted \( c \) units to the left of the graph of \( y = f(x) \).
- If \( c > 0 \), then the graph of \( y = f(x - c) \) is shifted \( c \) units to the right of the graph of \( y = f(x) \).
2.6 Exercises

Pictured below is the graph of a function $f$.

![Graph of function f]

The table that follows evaluates the function $f$ in the plot at key values of $x$. Notice the horizontal format, where the first point in the table is the ordered pair $(-6,0)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-6</th>
<th>-4</th>
<th>-2</th>
<th>0</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>-2</td>
<td>0</td>
</tr>
</tbody>
</table>

Use the graph and the table to complete each of the following tasks for Exercises 1-10.

i. Set up a coordinate system on graph paper. Label and scale each axis, then copy and label the original graph of $f$ onto your coordinate system. Remember to draw all lines with a ruler.

ii. Use the original table to help complete the table for the given function in the exercise.

iii. Using a different colored pencil, plot the data from your completed table on the same coordinate system as the original graph of $f$. Use these points to help complete the graph of the given function in the exercise, then label this graph with its equation given in the exercise.

1. $y = f(2x)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. $y = f((1/2)x)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-12</th>
<th>-8</th>
<th>-4</th>
<th>0</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. $y = f(-x)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-4</th>
<th>-2</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. $y = f(x + 3)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-9</th>
<th>-7</th>
<th>-5</th>
<th>-3</th>
<th>-1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. $y = f(x - 1)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-5</th>
<th>-3</th>
<th>-1</th>
<th>1</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6. $y = f(-2x)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
7. \( y = f\left((1/2)x\right) \).

\[
\begin{array}{c|cccccc}
 x & -8 & -4 & 0 & 4 & 8 & 12 \\
 y & & & & & & \\
\end{array}
\]

8. \( y = f(-x - 2) \).

\[
\begin{array}{c|cccccc}
 x & -6 & -4 & -2 & 0 & 2 & 4 \\
 y & & & & & & \\
\end{array}
\]

9. \( y = f(-x + 1) \).

\[
\begin{array}{c|cccccc}
 x & -3 & -1 & 1 & 3 & 5 & 7 \\
 y & & & & & & \\
\end{array}
\]

10. \( y = f(-x/4) \).

\[
\begin{array}{c|cccccc}
 x & -16 & -8 & 0 & 8 & 16 & 24 \\
 y & & & & & & \\
\end{array}
\]

11. Use your graphing calculator to draw the graph of \( y = \sqrt{x} \). Then, draw the graph of \( y = \sqrt{-x} \). In your own words, explain what you learned from this exercise.

12. Use your graphing calculator to draw the graph of \( y = |x| \). Then, draw the graph of \( y = |-x| \). In your own words, explain what you learned from this exercise.

13. Use your graphing calculator to draw the graph of \( y = x^2 \). Then, in succession, draw the graphs of \( y = (x - 2)^2 \), \( y = (x - 4)^2 \), and \( y = (x - 6)^2 \). In your own words, explain what you learned from this exercise.

14. Use your graphing calculator to draw the graph of \( y = x^2 \). Then, in succession, draw the graphs of \( y = (x + 2)^2 \), \( y = (x + 4)^2 \), and \( y = (x + 6)^2 \). In your own words, explain what you learned from this exercise.

15. Use your graphing calculator to draw the graph of \( y = |x| \). Then, in succession, draw the graphs of \( y = |2x| \), \( y = |3x| \), and \( y = |4x| \). In your own words, explain what you learned from this exercise.

16. Use your graphing calculator to draw the graph of \( y = |x| \). Then, in succession, draw the graphs of \( y = |(1/2)x| \), \( y = |(1/3)x| \), and \( y = |(1/4)x| \). In your own words, explain what you learned from this exercise.

Pictured below is the graph of a function \( f \). In Exercises 17-22, use this graph to perform each of the following tasks.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Make an exact copy of the graph of \( f \) on your coordinate system. Remember to draw all lines with a ruler.

ii. In the narrative, a shadow box at the end of the section summarizes the concepts and technique of horizontal scaling, horizontal reflection, and horizontal translation. Use the shortcut ideas presented in this summary shadow
box to draw the graphs of the functions that follow without using tables.

iii. Use a different colored pencil to draw the graph of the function given in the exercise. Label this graph with its equation. Be sure that key points are accurately plotted. In each exercise, please plot exactly two plots per coordinate system, the graph of original function $f$ and the graph of the function in the exercise.

17. $y = f(2x)$.
18. $y = f((1/2)x)$.
19. $y = f(x)$.
20. $y = f(x - 1)$.
21. $y = f(x + 3)$.
22. $y = f(x - 2)$.

Pictured below is the graph of a function $f$. In Exercises 23-28, use this graph to perform each of the following tasks.

23. $y = f(2x)$.
24. $y = f((1/2)x)$.
25. $y = f(x)$.
26. $y = f(x + 3)$.
27. $y = f(x - 2)$.
28. $y = f(x + 1)$.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Make an exact copy of the graph of $f$ on your coordinate system. Remember to draw all lines with a ruler.

ii. In the narrative, a shadow box at the end of the section summarizes the concepts and technique of horizontal scaling, horizontal reflection, and horizontal translation. Use the shortcut ideas presented in this summary shadow box to draw the graphs of the functions that follow without using tables.

iii. Use a different colored pencil to draw the graph of the function given in the exercise. Label this graph with its equation. Be sure that key points are accurately plotted. In each exercise, please plot exactly two plots per coordinate system, the graph of original function $f$ and the graph of the function in the exercise.
2.6 Answers

1. \[ y = f(2x) \]

3. \[ y = f(-x) \]

5. \[ y = f(x-1) \]

7. \[ y = f\left(-\frac{1}{2}x\right) \]
9. \[ y = f(-x+1) \]

11. Multiplying on the inside by \(-1\), as in \[ y = \sqrt{-x} \], reflects the graph across the y-axis.

13. Replacing \( x \) with \( x - c \), where \( c \) is positive, moves the graph \( c \) units to the right.

15. Multiplying by a scalar \( a \), such that \( a \) is larger than 1, compresses the graph horizontally by a factor of \( a \).

17. \[ y = f(2x) \]

19. \[ y = f(-x) \]

21. \[ y = f(x+3) \]

23. \[ y = f(2x) \]
25.\[ f(y) = f(-x) \]

27.\[ f(y) = f(x-2) \]
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Version: Fall 2007