7 Rational Functions

In this chapter, we begin our study of rational functions — functions of the form $p(x)/q(x)$, where $p$ and $q$ are both polynomials. Rational functions are similar in structure to rational numbers (commonly thought of as fractions), and they are studied and used extensively in mathematics, engineering, and science.

We will learn how to manipulate these functions, and discover the myriad algebraic tricks and pitfalls that accompany them. We will also see some of the ways that they can be applied to everyday situations, such as modeling the length of time it takes a group of people to complete a task, or calculating the distance traveled by an object.

In more advanced mathematics courses, such as college algebra and calculus, you will learn even more about the intricate nature of rational functions. In many science and engineering courses, you will use rational functions to model what you are studying. In your everyday life, you can use rational functions for a number of useful calculations, such as the amount of time or work that a given task might require. For these reasons, along with the fact that learning how to manipulate rational functions will further your understanding of mathematics, this chapter warrants a good deal of attention.

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7.1 Introducing Rational Functions

In the previous chapter, we studied polynomials, functions having equation form

\[ p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_n x^n. \] (1)

Even though this polynomial is presented in ascending powers of \( x \), the leading term of the polynomial is still \( a_n x^n \), the term with the highest power of \( x \). The degree of the polynomial is the highest power of \( x \) present, so in this case, the degree of the polynomial is \( n \).

In this section, our study will lead us to the rational functions. Note the root word “ratio” in the term “rational.” Does it remind you of the word “fraction”? It should, as rational functions are functions in a very specific fractional form.

**Definition 2.** A rational function is a function that can be written as a quotient of two polynomial functions. In symbols, the function

\[ f(x) = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m} \] (3)

is called a rational function.

For example,

\[ f(x) = \frac{1 + x}{x + 2}, \quad g(x) = \frac{x^2 - 2x - 3}{x + 4}, \quad \text{and} \quad h(x) = \frac{3 - 2x - x^2}{x^3 + 2x^2 - 3x - 5} \] (4)

are rational functions, while

\[ f(x) = \frac{1 + \sqrt{x}}{x^2 + 1}, \quad g(x) = \frac{x^2 + 2x - 3}{1 + x^{1/2} - 3x^2}, \quad \text{and} \quad h(x) = \sqrt{\frac{x^2 - 2x - 3}{x^2 + 4x - 12}} \] (5)

are not rational functions.

Each of the functions in equation (4) are rational functions, because in each case, the numerator and denominator of the given expression is a valid polynomial.

However, in equation (5), the numerator of \( f(x) \) is not a polynomial (polynomials do not allow the square root of the independent variable). Therefore, \( f \) is not a rational function.

Similarly, the denominator of \( g(x) \) in equation (5) is not a polynomial. Fractions are not allowed as exponents in polynomials. Thus, \( g \) is not a rational function.

Finally, in the case of function \( h \) in equation (5), although the radicand (the expression inside the radical) is a rational function, the square root prevents \( h \) from being a rational function.

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An important skill to develop is the ability to draw the graph of a rational function. Let’s begin by drawing the graph of one of the simplest (but most fundamental) rational functions.

**The Graph of** \( y = 1/x \)

In all new situations, when we are presented with an equation whose graph we’ve not considered or do not recognize, we begin the process of drawing the graph by creating a table of points that satisfy the equation. It’s important to remember that the graph of an equation is the set of all points that satisfy the equation. We note that zero is not in the domain of \( y = 1/x \) (division by zero makes no sense and is not defined), and create a table of points satisfying the equation shown in Figure 1.

![Figure 1](image)

**Figure 1.** At the right is a table of points satisfying the equation \( y = 1/x \). These points are plotted as solid dots on the graph at the left.

At this point (see Figure 1), it’s pretty clear what the graph is doing between \( x = -3 \) and \( x = -1 \). Likewise, it’s clear what is happening between \( x = 1 \) and \( x = 3 \). However, there are some open areas of concern.

1. What happens to the graph as \( x \) increases without bound? That is, what happens to the graph as \( x \) moves toward \( \infty \)?
2. What happens to the graph as \( x \) decreases without bound? That is, what happens to the graph as \( x \) moves toward \( -\infty \)?
3. What happens to the graph as \( x \) approaches zero from the right?
4. What happens to the graph as \( x \) approaches zero from the left?

Let’s answer each of these questions in turn. We’ll begin by discussing the “end-behavior” of the rational function defined by \( y = 1/x \). First, the right end. What happens as \( x \) increases without bound? That is, what happens as \( x \) increases toward \( \infty \)? In Table 1(a), we computed \( y = 1/x \) for \( x \) equalling 100, 1000, and 10000. Note how the \( y \)-values in Table 1(a) are all positive and approach zero.

Students in calculus use the following notation for this idea.

\[
\lim_{x \to \infty} y = \lim_{x \to \infty} \frac{1}{x} = 0 \tag{6}
\]
They say “the limit of $y$ as $x$ approaches infinity is zero.” That is, as $x$ approaches infinity, $y$ approaches zero.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = 1/x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.01</td>
</tr>
<tr>
<td>1000</td>
<td>0.001</td>
</tr>
<tr>
<td>10000</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

(a)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = 1/x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>−100</td>
<td>−0.01</td>
</tr>
<tr>
<td>−1000</td>
<td>−0.001</td>
</tr>
<tr>
<td>−10000</td>
<td>−0.0001</td>
</tr>
</tbody>
</table>

(b)

Table 1. Examining the end-behavior of $y = 1/x$.

A completely similar event happens at the left end. As $x$ decreases without bound, that is, as $x$ decreases toward $-\infty$, note that the $y$-values in Table 1(b) are all negative and approach zero. Calculus students have a similar notation for this idea.

$$\lim_{x \to -\infty} y = \lim_{x \to -\infty} \frac{1}{x} = 0. \quad (7)$$

They say “the limit of $y$ as $x$ approaches negative infinity is zero.” That is, as $x$ approaches negative infinity, $y$ approaches zero.

These numbers in Tables 1(a) and 1(b), and the ideas described above, predict the correct end-behavior of the graph of $y = 1/x$. At each end of the $x$-axis, the $y$-values must approach zero. This means that the graph of $y = 1/x$ must approach the $x$-axis for $x$-values at the far right- and left-ends of the graph. In this case, we say that the $x$-axis acts as a horizontal asymptote for the graph of $y = 1/x$. As $x$ approaches either positive or negative infinity, the graph of $y = 1/x$ approaches the $x$-axis. This behavior is shown in Figure 2.

![Figure 2. The graph of $1/x$ approaches the $x$-axis as $x$ increases or decreases without bound.](image)

Our last investigation will be on the interval from $x = -1$ to $x = 1$. Readers are again reminded that the function $y = 1/x$ is undefined at $x = 0$. Consequently, we will break this region in half, first investigating what happens on the region between $x = 0$...
and \( x = 1 \). We evaluate \( y = 1/x \) at \( x = 1/2 \), \( x = 1/4 \), and \( x = 1/8 \), as shown in the table in Figure 3, then plot the resulting points.

\[
\begin{array}{|c|c|}
\hline
x & y = 1/x \\
\hline
1/2 & 2 \\
1/4 & 4 \\
1/8 & 8 \\
\hline
\end{array}
\]

**Figure 3.** At the right is a table of points satisfying the equation \( y = 1/x \). These points are plotted as solid dots on the graph at the left.

Note that the \( x \)-values in the table in Figure 3 approach zero from the right, then note that the corresponding \( y \)-values are getting larger and larger. We could continue in this vein, adding points. For example, if \( x = 1/16 \), then \( y = 16 \). If \( x = 1/32 \), then \( y = 32 \). If \( x = 1/64 \), then \( y = 64 \). Each time we halve our value of \( x \), the resulting value of \( x \) is closer to zero, and the corresponding \( y \)-value doubles in size. Calculus students describe this behavior with the notation

\[
\lim_{x \to 0^+} y = \lim_{x \to 0^+} \frac{1}{x} = \infty.
\]

That is, as “\( x \) approaches zero from the right, the value of \( y \) grows to infinity.” This is evident in the graph in Figure 3, where we see the plotted points move closer to the vertical axis while at the same time moving upward without bound.

A similar thing happens on the other side of the vertical axis, as shown in Figure 4.

\[
\begin{array}{|c|c|}
\hline
x & y = 1/x \\
\hline
-1/2 & -2 \\
-1/4 & -4 \\
-1/8 & -8 \\
\hline
\end{array}
\]

**Figure 4.** At the right is a table of points satisfying the equation \( y = 1/x \). These points are plotted as solid dots on the graph at the left.
Again, calculus students would write
\[
\lim_{x \to 0^-} y = \lim_{x \to 0^-} \frac{1}{x} = -\infty.
\]
That is, "as \( x \) approaches zero from the left, the values of \( y \) decrease to negative infinity." In Figure 4, it is clear that as points move closer to the vertical axis (as \( x \) approaches zero) from the left, the graph decreases without bound.

The evidence gathered to this point indicates that the vertical axis is acting as a vertical asymptote. As \( x \) approaches zero from either side, the graph approaches the vertical axis, either rising to infinity, or falling to negative infinity. The graph cannot cross the vertical axis because the function is undefined there. The completed graph is shown in Figure 5.

The complete graph of \( y = 1/x \) in Figure 5 is called a hyperbola and serves as a fundamental starting point for all subsequent discussion in this section.

We noted earlier that the domain of the function defined by the equation \( y = 1/x \) is the set \( D = \{ x : x \neq 0 \} \). Zero is excluded from the domain because division by zero is undefined. It’s no coincidence that the graph has a vertical asymptote at \( x = 0 \). We’ll see this relationship reinforced in further examples.
TranslATIONS

In this section, we will translate the graph of \( y = \frac{1}{x} \) in both the horizontal and vertical directions.

**Example 10.** Sketch the graph of

\[
y = \frac{1}{x + 3} - 4.
\]

(11)

Technically, the function defined by \( y = 1/(x + 3) - 4 \) does not have the general form (3) of a rational function. However, in later chapters we will show how \( y = 1/(x + 3) - 4 \) can be manipulated into the general form of a rational function.

We know what the graph of \( y = 1/x \) looks like. If we replace \( x \) with \( x + 3 \), this will shift the graph of \( y = 1/x \) three units to the left, as shown in Figure 6(a). Note that the vertical asymptote has also shifted 3 units to the left of its original position (the y-axis) and now has equation \( x = -3 \). By tradition, we draw the vertical asymptote as a dashed line.

If we subtract 4 from the result in Figure 6(a), this will shift the graph in Figure 6(a) four units downward to produce the graph shown in Figure 6(b). Note that the horizontal asymptote also shifted 4 units downward from its original position (the x-axis) and now has equation \( y = -4 \).

If you examine equation (11), you note that you cannot use \( x = -3 \) as this will make the denominator of equation (11) equal to zero. In Figure 6(b), note that there is a vertical asymptote in the graph of equation (11) at \( x = -3 \). This is a common occurrence, which will be a central theme of this chapter.
Let’s ask another key question.

**Example 12.** What are the domain and range of the rational function presented in Example 10?

You can glance at the equation

\[ y = \frac{1}{x + 3} - 4 \]

of Example 10 and note that \( x = -3 \) makes the denominator zero and must be excluded from the domain. Hence, the domain of this function is \( D = \{ x : x \neq -3 \} \).

However, you can also determine the domain by examining the graph of the function in Figure 6(b). Note that the graph extends indefinitely to the left and right. One might first guess that the domain is all real numbers if it were not for the vertical asymptote at \( x = -3 \) interrupting the continuity of the graph. Because the graph of the function gets arbitrarily close to this vertical asymptote (on either side) without actually touching the asymptote, the graph does not contain a point having an \( x \)-value equaling \(-3\). Hence, the domain is as above, \( D = \{ x : x \neq -3 \} \). This is comforting that the graphical analysis agrees with our earlier analytical determination of the domain.

The graph is especially helpful in determining the range of the function. Note that the graph rises to positive infinity and falls to negative infinity. One would first guess that the range is all real numbers if it were not for the horizontal asymptote at \( y = -4 \) interrupting the continuity of the graph. Because the graph gets arbitrarily close to the horizontal asymptote (on either side) without actually touching the asymptote, the graph does not contain a point having a \( y \)-value equaling \(-4\). Hence, \(-4\) is excluded from the range. That is, \( R = \{ y : y \neq -4 \} \).

### Scaling and Reflection

In this section, we will both scale and reflect the graph of \( y = 1/x \). For extra measure, we also throw in translations in the horizontal and vertical directions.

**Example 13.** Sketch the graph of

\[ y = -\frac{2}{x - 4} + 3. \]  

(14)

First, we multiply the equation \( y = 1/x \) by \(-2\) to get

\[ y = -\frac{2}{x}. \]

Multiplying by \(2\) should stretch the graph in the vertical directions (both positive and negative) by a factor of \(2\). Note that points that are very near the \(x\)-axis, when doubled, are not going to stray too far from the \(x\)-axis, so the horizontal asymptote will remain the same. Finally, multiplying by \(-2\) will not only stretch the graph, it will also reflect the graph across the \(x\)-axis, as shown in Figure 7(b).²

² Recall that we saw similar behavior when studying the parabola. The graph of \( y = -2x^2 \) stretched (vertically) the graph of the equation \( y = x^2 \) by a factor of \(2\), then reflected the result across the \(x\)-axis.
Replacing $x$ with $x - 4$ will shift the graph 4 units to the right, then adding 3 will shift the graph 3 units up, as shown in Figure 8. Note again that $x = 4$ makes the denominator of $y = -2/(x - 4) + 3$ equal to zero and there is a vertical asymptote at $x = 4$. The domain of this function is $D = \{x : x \neq 4\}$.

As $x$ approaches positive or negative infinity, points on the graph of $y = -2/(x - 4) + 3$ get arbitrarily close to the horizontal asymptote $y = 3$ but never touch it. Therefore, there is no point on the graph that has a $y$-value of 3. Thus, the range of the function is the set $R = \{y : y \neq 3\}$. 

**Figure 7.** Scaling and reflecting the graph of $y = 1/x$.

**Figure 8.** The graph of $y = -2/(x - 4) + 3$ is shifted 4 units right and 3 units up.

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Difficulties with the Graphing Calculator

The graphing calculator does a very good job drawing the graphs of “continuous functions.”

A continuous function is one that can be drawn in one continuous stroke, never lifting pen or pencil from the paper during the drawing.

Polynomials, such as the one in Figure 9, are continuous functions.

Unfortunately, a rational function with vertical asymptote(s) is not a continuous function. First, you have to lift your pen at points where the denominator is zero, because the function is undefined at these points. Secondly, it’s not uncommon to have to jump from positive infinity to negative infinity (or vice-versa) when crossing a vertical asymptote. When this happens, we have to lift our pen and shift it before continuing with our drawing.

However, the graphing calculator does not know how to do this “lifting” of the pen near vertical asymptotes. The graphing calculator only knows one technique, plot a point, then connect it with a segment to the last point plotted, move an incremental distance and repeat. Consequently, when the graphing calculator crosses a vertical asymptote where there is a shift from one type of infinity to another (e.g., from positive to negative), the calculator draws a “false line” of connection, one that it should not draw. Let’s demonstrate this aberration with an example.

Example 15. Use a graphing calculator to draw the graph of the rational function in Example 13.

Load the equation into your calculator, as shown in Figure 10(a). Set the window as shown in Figure 10(b), then push the GRAPH button to draw the graph shown in Figure 10(c). Results may differ on some calculators, but in our case, note the “false
"line" drawn from the top of the screen to the bottom, attempting to “connect” the two branches of the hyperbola.

Some might rejoice and claim, “Hey, my graphing calculator draws vertical asymptotes.” However, before you get too excited, note that in Figure 8 the vertical asymptote should occur at exactly $x = 4$. If you look very carefully at the “vertical line” in Figure 10(c), you’ll note that it just misses the tick mark at $x = 4$. This “vertical line” is a line that the calculator should not draw. The calculator is attempting to draw a continuous function where one doesn’t exist.

![Figure 10.](image)
Figure 10. The calculator attempts to draw a continuous function when it shouldn’t.

One possible workaround\(^3\) is to press the MODE button on your keyboard, which opens the menu shown in Figure 11(a). Use the arrow keys to highlight DOT instead of CONNECTED and press the ENTER key to make the selection permanent. Press the GRAPH button to draw the graph in Figure 11(b).

![Figure 11.](image)
Figure 11. The same graph in “dot mode.”

This “dot mode” on your calculator calculates the next point on the graph and plots the point, but it does not connect it with a line segment to the previously plotted point. This mode is useful in demonstrating that the vertical line in Figure 10(c) is not really part of the graph, but we lose some parts of the graph we’d really like to see. Compromise is in order.

This example clearly shows that intelligent use of the calculator is a required component of this course. The calculator is not simply a “black box” that automatically does what you want it to do. In particular, when you are drawing rational functions, it helps to know ahead of time the placement of the vertical asymptotes. Knowledge

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\(^3\) Instructors might discuss a number of alternative strategies to represent rational functions on the graphing calculator. What we present here is only one of a number of approaches.
of the asymptotes, coupled with what you see on your calculator screen, should enable you to draw a graph as accurate as that shown in Figure 8.

**Gentle reminder.** You’ll want to set your calculator back in “connected mode.” To do this, press the **MODE** button on your keyboard to open the menu in Figure 10(a) once again. Use your arrow keys to highlight **CONNECTED**, then press the **ENTER** key to make the selection permanent.
7.1 Exercises

In Exercises 1-14, perform each of the following tasks for the given rational function.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis.

ii. Use geometric transformations as in Examples 10, 12, and 13 to draw the graphs of each of the following rational functions. Draw the vertical and horizontal asymptotes as dashed lines and label each with its equation. You may use your calculator to check your solution, but you should be able to draw the rational function without the use of a calculator.

iii. Use set-builder notation to describe the domain and range of the given rational function.

1. \( f(x) = \frac{-2}{x} \)
2. \( f(x) = \frac{3}{x} \)
3. \( f(x) = \frac{1}{x - 4} \)
4. \( f(x) = \frac{1}{x + 3} \)
5. \( f(x) = \frac{2}{x - 5} \)
6. \( f(x) = \frac{-3}{x + 6} \)
7. \( f(x) = \frac{1}{x - 2} \)
8. \( f(x) = \frac{-1}{x + 4} \)
9. \( f(x) = \frac{-2}{x - 5} \)
10. \( f(x) = \frac{3}{x - 5} \)
11. \( f(x) = \frac{1}{x - 2} - 3 \)
12. \( f(x) = \frac{-1}{x + 1} + 5 \)
13. \( f(x) = \frac{-2}{x - 3} - 4 \)
14. \( f(x) = \frac{3}{x + 5} - 2 \)

In Exercises 15-22, find all vertical asymptotes, if any, of the graph of the given function.

15. \( f(x) = \frac{-5}{x + 1} - 3 \)
16. \( f(x) = \frac{6}{x + 8} + 2 \)
17. \( f(x) = \frac{-9}{x + 2} - 6 \)
18. \( f(x) = \frac{-8}{x - 4} - 5 \)
19. \( f(x) = \frac{2}{x + 5} + 1 \)
20. \( f(x) = \frac{-3}{x + 9} + 2 \)
21. \( f(x) = \frac{7}{x + 8} - 9 \)
22. \( f(x) = \frac{6}{x - 5} - 8 \)

In Exercises 23-30, find all horizontal asymptotes, if any, of the graph of the given function.

23. \( f(x) = \frac{5}{x + 7} + 9 \)
24. \( f(x) = \frac{-8}{x + 7} - 4 \)

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25. \( f(x) = \frac{8}{x+5} - 1 \)

26. \( f(x) = -\frac{2}{x+3} + 8 \)

27. \( f(x) = \frac{7}{x+1} - 9 \)

28. \( f(x) = -\frac{2}{x-1} + 5 \)

29. \( f(x) = \frac{5}{x+2} - 4 \)

30. \( f(x) = -\frac{6}{x-1} - 2 \)

In Exercises 31-38, state the domain of the given rational function using set-builder notation.

31. \( f(x) = \frac{4}{x+5} + 5 \)

32. \( f(x) = -\frac{7}{x-6} + 1 \)

33. \( f(x) = \frac{6}{x-5} + 1 \)

34. \( f(x) = -\frac{5}{x-3} - 9 \)

35. \( f(x) = \frac{1}{x+7} + 2 \)

36. \( f(x) = -\frac{2}{x-5} + 4 \)

37. \( f(x) = -\frac{4}{x+2} + 2 \)

38. \( f(x) = \frac{2}{x+6} + 9 \)

In Exercises 39-46, find the range of the given function, and express your answer in set notation.

39. \( f(x) = \frac{2}{x-3} + 8 \)

40. \( f(x) = \frac{4}{x-3} + 5 \)

41. \( f(x) = -\frac{5}{x-8} - 5 \)

42. \( f(x) = -\frac{2}{x+1} + 6 \)

43. \( f(x) = \frac{7}{x+7} + 5 \)

44. \( f(x) = -\frac{8}{x+3} + 9 \)

45. \( f(x) = \frac{4}{x+3} - 2 \)

46. \( f(x) = -\frac{5}{x-4} + 9 \)
7.1 **Answers**

1. $D = \{x : x \neq 0\}, R = \{y : y \neq 0\}$

3. $D = \{x : x \neq 4\}, R = \{y : y \neq 0\}$

5. $D = \{x : x \neq 5\}, R = \{y : y \neq 0\}$

7. $D = \{x : x \neq 0\}, R = \{y : y \neq -2\}$

9. $D = \{x : x \neq 0\}, R = \{y : y \neq -5\}$

11. $D = \{x : x \neq 2\}, R = \{y : y \neq -3\}$
13. \[ D = \{ x : x \neq 3 \}, \quad R = \{ y : y \neq -4 \} \]

15. Vertical asymptote: \( x = -1 \)

17. Vertical asymptote: \( x = -2 \)

19. Vertical asymptote: \( x = -5 \)

21. Vertical asymptote: \( x = -8 \)

23. Horizontal asymptote: \( y = 9 \)

25. Horizontal asymptote: \( y = -1 \)

27. Horizontal asymptote: \( y = -9 \)

29. Horizontal asymptote: \( y = -4 \)

31. Domain = \( \{ x : x \neq -5 \} \)

33. Domain = \( \{ x : x \neq 5 \} \)

35. Domain = \( \{ x : x \neq -7 \} \)

37. Domain = \( \{ x : x \neq -2 \} \)

39. Range = \( \{ y : y \neq 8 \} \)

41. Range = \( \{ y : y \neq -5 \} \)

43. Range = \( \{ y : y \neq 5 \} \)

45. Range = \( \{ y : y \neq -2 \} \)
7.2 Reducing Rational Functions

The goal of this section is to learn how to reduce a rational expression to “lowest terms.” Of course, that means that we will have to understand what is meant by the phrase “lowest terms.” With that thought in mind, we begin with a discussion of the greatest common divisor of a pair of integers.

First, we define what we mean by “divisibility.”

**Definition 1.** Suppose that we have a pair of integers $a$ and $b$. We say that “$a$ is a divisor of $b$,” or “$a$ divides $b$” if and only if there is another integer $k$ so that $b = ak$. Another way of saying the same thing is to say that $a$ divides $b$ if, upon dividing $b$ by $a$, the remainder is zero.

Let’s look at an example.

**Example 2.** What are the divisors of 12?

Because $12 = 1 \times 12$, both 1 and 12 are divisors of 12. Because $12 = 2 \times 6$, both 2 and 6 are divisors of 12. Finally, because $12 = 3 \times 4$, both 3 and 4 are divisors of 12. If we list them in ascending order, the divisors of 12 are

$$1, 2, 3, 4, 6, \text{ and } 12.$$  

Let’s look at another example.

**Example 3.** What are the divisors of 18?

Because $18 = 1 \times 18$, both 1 and 18 are divisors of 18. Similarly, $18 = 2 \times 9$ and $18 = 3 \times 6$, so in ascending order, the divisors of 18 are

$$1, 2, 3, 6, 9, \text{ and } 18.$$  

The greatest common divisor of two or more integers is the largest divisor the integers share in common. An example should make this clear.

**Example 4.** What is the greatest common divisor of 12 and 18?

In **Example 2** and **Example 3**, we saw the following.

Divisors of 12: $1, 2, 3, 4, 6, 12$

Divisors of 18: $1, 2, 3, 6, 9, 18$

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6 The word “divisor” and the word “factor” are synonymous.
We’ve framed the divisors that 12 and 18 have in common. They are 1, 2, 3, and 6. The “greatest” of these “common” divisors is 6. Hence, we say that “the greatest common divisor of 12 and 18 is 6.”

**Definition 5.** The greatest common divisor of two integers \( a \) and \( b \) is the largest divisor they have in common. We will use the notation 
\[
GCD(a, b)
\]
to represent the greatest common divisor of \( a \) and \( b \).

Thus, as we saw in Example 4, \( GCD(12, 18) = 6 \).

When the greatest common divisor of a pair of integers is one, we give that pair a special name.

**Definition 6.** Let \( a \) and \( b \) be integers. If the greatest common divisor of \( a \) and \( b \) is one, that is, if \( GCD(a, b) = 1 \), then we say that \( a \) and \( b \) are relatively prime.

For example:

- 9 and 12 are not relatively prime because \( GCD(9, 12) = 3 \).
- 10 and 15 are not relatively prime because \( GCD(10, 15) = 5 \).
- 8 and 21 are relatively prime because \( GCD(8, 21) = 1 \).

We can now define what is meant when we say that a rational number is reduced to lowest terms.

**Definition 7.** A rational number in the form \( p/q \), where \( p \) and \( q \) are integers, is said to be reduced to lowest terms if and only if \( GCD(p, q) = 1 \). That is, \( p/q \) is reduced to lowest terms if the greatest common divisor of both numerator and denominator is 1.

As we saw in Example 4, the greatest common divisor of 12 and 18 is 6. Therefore, the fraction \( 12/18 \) is not reduced to lowest terms. However, we can reduce \( 12/18 \) to lowest terms by dividing both numerator and denominator by their greatest common divisor. That is,
\[
\frac{12}{18} = \frac{12 \div 6}{18 \div 6} = \frac{2}{3}
\]
Note that \( GCD(2, 3) = 1 \), so \( 2/3 \) is reduced to lowest terms.
When it is difficult to ascertain the greatest common divisor, we’ll find it more efficient to proceed as follows:

- Prime factor both numerator and denominator.
- Cancel common factors.

Thus, to reduce $\frac{12}{18}$ to lowest terms, first express both numerator and denominator as a product of prime numbers, then cancel common primes.

\[
\frac{12}{18} = \frac{2 \cdot 2 \cdot 3}{2 \cdot 3 \cdot 3} = \frac{2 \cdot 2 \cdot 3}{2 \cdot 3 \cdot 3} = \frac{2}{3} \tag{8}
\]

When you cancel a 2, you’re actually dividing both numerator and denominator by 2. When you cancel a 3, you’re actually dividing both numerator and denominator by 3. Note that doing both (dividing by 2 and then dividing by 3) is equivalent to dividing both numerator and denominator by 6.

We will favor this latter technique, precisely because it is identical to the technique we will use to reduce rational functions to lowest terms. However, this “cancellation” technique has some pitfalls, so let’s take a moment to discuss some common cancellation mistakes.

**Cancellation**

You can spark some pretty heated debate amongst mathematics educators by innocently mentioning the word “cancellation.” There seem to be two diametrically opposed camps, those who don’t mind when their students use the technique of cancellation, and on the other side, those that refuse to even use the term “cancellation” in their classes.

Both sides of the argument have merit. As we showed in equation (8), we can reduce $\frac{12}{18}$ quite efficiently by simply canceling common factors. On the other hand, instructors from the second camp prefer to use the phrase “factor out a 1” instead of the phrase “cancel,” encouraging their students to reduce $\frac{12}{18}$ as follows.

\[
\frac{12}{18} = \frac{2 \cdot 2 \cdot 3}{2 \cdot 3 \cdot 3} = \frac{2 \cdot 2 \cdot 3}{2 \cdot 3 \cdot 3} = \frac{2}{3} \cdot 1 = \frac{2}{3}
\]

This is a perfectly valid technique and one that, quite honestly, avoids the quicksand of “cancellation mistakes.” Instructors who grow weary of watching their students “cancel” when they shouldn’t are quite likely to promote this latter technique.

However, if we can help our students avoid “cancellation mistakes,” we prefer to allow our students to cancel common factors (as we did in equation (8)) when reducing fractions such as $\frac{12}{18}$ to lowest terms. So, with these thoughts in mind, let’s discuss some of the most common cancellation mistakes.

Let’s begin with a most important piece of advice.

**How to Avoid Cancellation Mistakes.** You may only cancel factors, not addends. To avoid cancellation mistakes, factor **completely** before you begin to cancel.
Warning 9. Many of the ensuing calculations are incorrect. They are examples of common mistakes that are made when performing cancellation. Make sure that you read carefully and avoid just “scanning” these calculations.

As a first example, consider the rational expression
\[ \frac{2 + 6}{2}, \]
which clearly equals 8/2, or 4. However, if you cancel in this situation, as in
\[ \frac{2 + 6}{2} = \frac{2 + 6}{2}, \]
(10)
you certainly do not get the same result. So, what happened?

Note that in the numerator of equation (10), the 2 and the 6 are separated by a plus sign. Thus, they are not factors; they are addends! You are not allowed to cancel addends, only factors.

Suppose, for comparison, that the rational expression had been
\[ \frac{2 \cdot 6}{2}, \]
which clearly equals 12/2, or 6. In this case, the 2 and the 6 in the numerator are separated by a multiplication symbol, so they are factors and cancellation is allowed, as in
\[ \frac{2 \cdot 6}{2} = \frac{2 \cdot 6}{2} = 6. \]
(11)

Now, before you dismiss these examples as trivial, consider the following examples which are identical in structure. First, consider
\[ \frac{x + (x + 2)}{x} = \frac{x + (x + 2)}{x} = x + 2. \]
This cancellation is identical to that performed in equation (10) and is not allowed. In the numerator, note that \( x \) and \( x + 2 \) are separated by an addition symbol, so they are addends. You are not allowed to cancel addends! 

Conversely, consider the following example.
\[ \frac{x(x + 2)}{x} = \frac{x(x + 2)}{x} = x + 2 \]
In the numerator of this example, \( x \) and \( x + 2 \) are separated by implied multiplication. Hence, they are factors and cancellation is permissible.

Look again at equation (10), where the correct answer should have been 8/2, or 4. We mistakenly found the answer to be 6, because we cancelled addends. A workaround would be to first factor the numerator of equation (10), then cancel, as follows.
\[
\frac{2 + 6}{2} = \frac{2(1 + 3)}{2} = 2(1 + 3) = 1 + 3 = 4
\]

Note that we cancelled factors in this approach, which is permissible, and got the correct answer 4.

**Warning 12.** We are finished discussing common cancellation mistakes and you may not continue reading with confidence that all mathematics is correctly presented.

### Reducing Rational Expressions in \(x\)

Now that we’ve discussed some fundamental ideas and techniques, let’s apply what we’ve learned to rational expressions that are functions of an independent variable (usually \(x\)). Let’s start with a simple example.

**Example 13.** Reduce the rational expression

\[
\frac{2x - 6}{x^2 - 7x + 12}
\]

(14)

to lowest terms. For what values of \(x\) is your result valid?

In the numerator, factor out a 2, as in \(2x - 6 = 2(x - 3)\).

The denominator is a quadratic trinomial with \(ac = (1)(12) = 12\). The integer pair \(-3\) and \(-4\) has product 12 and sum \(-7\), so the denominator factors as shown.

\[
\frac{2x - 6}{x^2 - 7x + 12} = \frac{2(x - 3)}{(x - 3)(x - 4)}.
\]

Now that both numerator and denominator are factored, we can cancel common factors.

\[
\frac{2x - 6}{x^2 - 7x + 12} = \frac{2(x - 3)}{(x - 3)(x - 4)} = \frac{2}{x - 4}
\]

Thus, we have shown that

\[
\frac{2x - 6}{x^2 - 7x + 12} = \frac{2}{x - 4}.
\]

(15)

In equation (15), we are stating that the expression on the left (the original expression) is identical to the expression on the right for all values of \(x\).

Actually, there are two notable exceptions, the first of which is \(x = 3\). If we substitute \(x = 3\) into the left-hand side of equation (15), we get

\[
\frac{2x - 6}{x^2 - 7x + 12} = \frac{2(3) - 6}{(3)^2 - 7(3) + 12} = 0
\]

We cannot divide by zero, so the left-hand side of equation (15) is undefined if \(x = 3\). Therefore, the result in equation (15) is not valid if \(x = 3\).
Similarly, if we insert \( x = 4 \) in the left-hand side of equation (15),

\[
\frac{2x - 6}{x^2 - 7x + 12} = \frac{2(4) - 6}{(4)^2 - 7(4) + 12} = \frac{2}{0}.
\]

Again, division by zero is undefined. The left-hand side of equation (15) is undefined if \( x = 4 \), so the result in equation (15) is not valid if \( x = 4 \). Note that the right-hand side of equation (15) is also undefined at \( x = 4 \).

However, the algebraic work we did above guarantees that the left-hand side of equation (15) will be identical to the right-hand side of equation (15) for all other values of \( x \). For example, if we substitute \( x = 5 \) into the left-hand side of equation (15),

\[
\frac{2x - 6}{x^2 - 7x + 12} = \frac{2(5) - 6}{(5)^2 - 7(5) + 12} = \frac{4}{2} = 2.
\]

On the other hand, if we substitute \( x = 5 \) into the right-hand side of equation (15),

\[
\frac{2}{x - 4} = \frac{2}{5 - 4} = 2.
\]

Hence, both sides of equation (15) are identical when \( x = 5 \). In a similar manner, we could check the validity of the identity in equation (15) for all other values of \( x \).

You can use the graphing calculator to verify the identity in equation (15). Load the left- and right-hand sides of equation (15) in Y= menu, as shown in Figure 1(a). Press 2nd TBLSET and adjust settings as shown in Figure 1(b). Be sure that you highlight AUTO for both independent and dependent variables and press ENTER on each to make the selection permanent. In Figure 1(b), note that we’ve set TblStart = 0 and \( \Delta \text{Tbl} = 1 \). Press 2nd TABLE to produce the tabular results shown in Figure 1(c).

![Figure 1](Image)

(a) (b) (c)

Figure 1. Using the graphing calculator to check that the left- and right-hand sides of equation (15) are identical.

Remember that we placed the left- and right-hand sides of equation (15) in Y1 and Y2, respectively.

- In the tabular results of Figure 1(c), note the ERR (error) message in Y1 when \( x = 3 \) and \( x = 4 \). This agrees with our findings above, where the left-hand side of equation (15) was undefined because of the presence of zero in the denominator when \( x = 3 \) or \( x = 4 \).
- In the tabular results of Figure 1(c), note that the value of Y1 and Y2 agree for all other values of \( x \).
We are led to the following key result.

**Restrictions.** In general, when you reduce a rational expression to lowest terms, the expression obtained should be **identical** to the original expression for all values of the variables in each expression, save those values of the variables that make any denominator equal to zero. This applies to the denominator in the original expression, all intermediate expressions in your work, and the final result. We will refer to any values of the variable that make any denominator equal to zero as **restrictions**.

Let’s look at another example.

**Example 16.** Reduce the expression

\[
\frac{2x^2 + 5x - 12}{4x^3 + 16x^2 - 9x - 36} \tag{17}
\]

to lowest terms. State all restrictions.

The numerator is a quadratic trinomial with \(ac = (2)(-12) = -24\). The integer pair \(-3\) and \(8\) have product \(-24\) and sum \(5\). Break the middle term of the polynomial in the numerator into a sum using this integer pair, then factor by grouping.

\[
2x^2 + 5x - 12 = 2x^2 - 3x + 8x - 12 = x(2x - 3) + 4(2x - 3) = (x + 4)(2x - 3)
\]

Factor the denominator by grouping.

\[
4x^3 + 16x^2 - 9x - 36 = 4x^2(x + 4) - 9(x + 4) = (4x^2 - 9)(x + 4) = (2x + 3)(2x - 3)(x + 4)
\]

Note how the difference of two squares pattern was used to factor \(4x^2 - 9 = (2x + 3)(2x - 3)\) in the last step.

Now that we’ve factored both numerator and denominator, we cancel common factors.

\[
\frac{2x^2 + 5x - 12}{4x^3 + 16x^2 - 9x - 36} = \frac{(x + 4)(2x - 3)}{(2x + 3)(2x - 3)(x + 4)} = \frac{(x + 4)(2x - 3)}{(2x + 3)(2x - 3)(x + 4)} = \frac{1}{2x + 3}
\]

We must now determine the restrictions. This means that we must find those values of \(x\) that make any denominator equal to zero.
In the body of our work, we have the denominator \((2x + 3)(2x - 3)(x + 4)\). If we set this equal to zero, the zero product property implies that

\[
2x + 3 = 0 \quad \text{or} \quad 2x - 3 = 0 \quad \text{or} \quad x + 4 = 0.
\]

Each of these linear factors can be solved independently.

\[
x = -\frac{3}{2} \quad \text{or} \quad x = \frac{3}{2} \quad \text{or} \quad x = -4
\]

Each of these \(x\)-values is a restriction.

In the final rational expression, the denominator is \(2x + 3\). This expression equals zero when \(x = -\frac{3}{2}\) and provides no new restrictions.

Because the denominator of the original expression, namely \(4x^3 + 16x^2 - 9x - 36\), is identical to its factored form in the body of our work, this denominator will produce no new restrictions.

Thus, for all values of \(x\),

\[
\frac{2x^2 + 5x - 12}{4x^3 + 16x^2 - 9x - 36} = \frac{1}{2x + 3},
\]

provided \(x \neq -3/2, 3/2, \) or \(-4\). These are the restrictions. The two expressions are identical for all other values of \(x\).

Finally, let’s check this result with our graphing calculator. Load each side of equation (18) into the Y= menu, as shown in Figure 2(a). We know that we have a restriction at \(x = -3/2\), so let’s set TblStart = -2 and \(\Delta Tbl = 0.5\), as shown in Figure 2(b). Be sure that you have AUTO set for both independent and dependent variables. Push the TABLE button to produce the tabular display shown in Figure 2(c).

![Figure 2](image)

Figure 2. Using the graphing calculator to check that the left- and right-hand sides of equation (18) are identical.

Remember that we placed the left- and right-hand sides of equation (18) in \(Y1\) and \(Y2\), respectively.

- In Figure 2(c), note that the expressions \(Y1\) and \(Y2\) agree at all values of \(x\) except \(x = -1.5\). This is the restriction \(-3/2\) we found above.
- Use the down arrow key to scroll down in the table shown in Figure 2(c) to produce the tabular view shown in Figure 2(d). Note that \(Y1\) and \(Y2\) agree for all values of \(x\) except \(x = 1.5\). This is the restriction \(3/2\) we found above.
- We leave it to our readers to uncover the restriction at \(x = -4\) by using the up-arrow to scroll up in the table until you reach an \(x\)-value of \(-4\). You should uncover
another ERR (error) message at this $x$-value because it is a restriction. You get the ERR message due to the fact that the denominator of the left-hand side of equation (18) is zero at $x = -4$.

**Sign Changes**

It is not uncommon that you will have to manipulate the signs in a fraction in order to obtain common factors that can be then cancelled. Consider, for example, the rational expression

$$\frac{3 - x}{x - 3}. \tag{19}$$

One possible approach is to factor $-1$ out of the numerator to obtain

$$\frac{3 - x}{x - 3} = \frac{-(x - 3)}{x - 3}.$$  

You can now cancel common factors.$^7$

$$\frac{3 - x}{x - 3} = \frac{-(x - 3)}{x - 3} = \frac{-(x - 3)}{x - 3} = -1$$

This result is valid for all values of $x$, provided $x \neq 3$.

Let’s look at another example.

**Example 20.** Reduce the rational expression

$$\frac{2x - 2x^3}{3x^3 + 4x^2 - 3x - 4} \tag{21}$$

to lowest terms. State all restrictions.

In the numerator, factor out $2x$, then complete the factorization using the difference of two squares pattern.

$$2x - 2x^3 = 2x(1 - x^2) = 2x(1 + x)(1 - x)$$

The denominator can be factored by grouping.

$$3x^3 + 4x^2 - 3x - 4 = x^2(3x + 4) - 1(3x + 4) = (x^2 - 1)(3x + 4)$$

$$= (x + 1)(x - 1)(3x + 4)$$

Note how the difference of two squares pattern was applied in the last step.

---

$^7$ When everything cancels, the resulting rational expression equals 1. For example, consider $\frac{6}{6}$, which surely is equal to 1. If we factor and cancel common factors, everything cancels.

$$\frac{6}{6} = \frac{2 \cdot 3}{2 \cdot 3} = \frac{\cancel{2} \cdot \cancel{3}}{\cancel{2} \cdot \cancel{3}} = 1$$

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At this point,
\[
\frac{2x - 2x^3}{3x^3 + 4x^2 - 3x - 4} = \frac{2x(1 + x)(1 - x)}{(x + 1)(x - 1)(3x + 4)}.
\]
Because we have \(1 - x\) in the numerator and \(x - 1\) in the denominator, we will factor out a \(-1\) from \(1 - x\), and because the order of factors does not affect their product, we will move the \(-1\) out to the front of the numerator.

\[
\frac{2x - 2x^3}{3x^3 + 4x^2 - 3x - 4} = \frac{2x(1 + x)(-1)(x - 1)}{(x + 1)(x - 1)(3x + 4)} = -2x(1 + x)(x - 1)
\]

We can now cancel common factors.

\[
\frac{2x - 2x^3}{3x^3 + 4x^2 - 3x - 4} = \frac{-2x(1 + x)(x - 1)}{(x + 1)(3x + 4)} = \frac{-2x}{3x + 4}
\]

Note that \(x + 1\) is identical to \(1 + x\) and cancels. Thus,

\[
\frac{2x - 2x^3}{3x^3 + 4x^2 - 3x - 4} = \frac{-2x}{3x + 4} \tag{22}
\]

for all values of \(x\), provided \(x \neq -1, 1, \text{ or } -4/3\). These are the restrictions, values of \(x\) that make denominators equal to zero.

---

**The Sign Change Rule for Fractions**

Let’s look at an alternative approach to the last example. First, let’s share the precept that every fraction has three signs, one on the numerator, one on the denominator, and a third on the fraction bar. Thus,

\[
\frac{-2}{3} \quad \text{has understood signs} \quad + \frac{-2}{+3}.
\]

Let’s state the *sign change rule* for fractions.

---

**The Sign Change Rule for Fractions.** Every fraction has three signs, one on the numerator, one on the denominator, and one on the fraction bar. If you don’t see an explicit sign, then a plus sign is understood. If you negate any two of these parts,

- numerator and denominator, or
- numerator and fraction bar, or
- fraction bar and denominator,

then the fraction remains unchanged.
For example, let’s start with $-\frac{2}{3}$, then do two negations: numerator and fraction bar. Then,

$$\frac{-2}{3} = \frac{+2}{+3}, \quad \text{or with understood plus signs,} \quad \frac{-2}{3} = \frac{-2}{-3}.$$ 

This is a familiar result, as negative two divided by a positive three equals a negative two-thirds.

On another note, we might decide to negate numerator and denominator. Then $-\frac{2}{3}$ becomes

$$\frac{+2}{+3} = \frac{+2}{3}, \quad \text{or with understood plus signs,} \quad \frac{-2}{3} = \frac{2}{-3}.$$ 

Again, a familiar result. Certainly, negative two divided by positive three is the same as positive two divided by negative three. They both equal minus two-thirds.

So there you have it. Negate any two parts of a fraction and it remains unchanged. On the surface, this seems a trivial remark, but it can be put to good use when reducing rational expressions. Suppose, for example, that we take the original rational expression from Example 20 and negate the numerator and fraction bar.

$$\frac{2x - 2x^3}{3x^3 + 4x^2 - 3x - 4} = -\frac{2x^3 - 2x}{3x^3 + 4x^2 - 3x - 4}$$

Note how we’ve made two sign changes. We’ve negated the fraction bar, we’ve negated the numerator ($-(2x - 2x^3) = 2x^3 - 2x$), and left the denominator alone. Therefore, the fraction is unchanged according to our sign change rule.

Now, factor and cancel common factors (we leave the steps for our readers — they’re similar to those we took in Example 20).

$$\frac{2x - 2x^3}{3x^3 + 4x^2 - 3x - 4} = -\frac{2x^3 - 2x}{3x^3 + 4x^2 - 3x - 4} = -\frac{2x(x + 1)(x - 1)}{(x + 1)(x - 1)(3x + 4)} = -\frac{2x}{3x + 4}$$

But does this answer match the answer in equation (22)? It does, as can be seen by making two negations, fraction bar and numerator.

$$-\frac{2x}{3x + 4} = -\frac{2x}{3x + 4}$$
The Secant Line

Consider the graph of the function \( f \) that we’ve drawn in Figure 3. Note that we’ve chosen two points on the graph of \( f \), namely \((a, f(a))\) and \((x, f(x))\), and we’ve drawn a line \( L \) through them that mathematicians call the “secant line.”

![Figure 3. The secant line passes through \((a, f(a))\) and \((x, f(x))\).](image)

The slope of the secant line \( L \) is found by dividing the change in \( y \) by the change in \( x \).

\[
\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}
\]  

(23)

This slope provides the average rate of change of the variable \( y \) with respect to the variable \( x \). Students in calculus use this “average rate of change” to develop the notion of “instantaneous rate of change.” However, we’ll leave that task for the calculus students and concentrate on the challenge of simplifying the expression equation (23) for the average rate of change.

Example 24. Given the function \( f(x) = x^2 \), simplify the expression for the average rate of change, namely

\[
\frac{f(x) - f(a)}{x - a}
\]

First, note that \( f(x) = x^2 \) and \( f(a) = a^2 \), so we can write

\[
\frac{f(x) - f(a)}{x - a} = \frac{x^2 - a^2}{x - a}.
\]

We can now use the difference of two squares pattern to factor the numerator and cancel common factors.

\[
\frac{x^2 - a^2}{x - a} = \frac{(x + a)(x - a)}{x - a} = x + a
\]
Thus,

\[
\frac{f(x) - f(a)}{x - a} = x + a,
\]

provided, of course, that \( x \neq a \).

Let’s look at another example.

**Example 25.** Consider the function \( f(x) = x^2 - 3x - 5 \). Simplify

\[
\frac{f(x) - f(2)}{x - 2}.
\]

First, \( f(x) = x^2 - 3x - 5 \) and therefore \( f(2) = (2)^2 - 3(2) - 5 = -7 \), so we can write

\[
\frac{f(x) - f(2)}{x - 2} = \frac{(x^2 - 3x - 5) - (-7)}{x - 2} = \frac{x^2 - 3x + 2}{x - 2}.
\]

We can now factor the numerator and cancel common factors.

\[
\frac{x^2 - 3x + 2}{x - 2} = \frac{(x - 2)(x - 1)}{x - 2} = x - 1
\]

Thus,

\[
\frac{f(x) - f(2)}{x - 2} = x - 1,
\]

provided, of course, that \( x \neq 2 \).
7.2 Exercises

In Exercises 1-12, reduce each rational number to lowest terms by applying the following steps:

i. Prime factor both numerator and denominator.
ii. Cancel common prime factors.
iii. Simplify the numerator and denominator of the result.

1. \( \frac{147}{98} \)
2. \( \frac{3087}{245} \)
3. \( \frac{1715}{196} \)
4. \( \frac{225}{50} \)
5. \( \frac{1715}{441} \)
6. \( \frac{56}{24} \)
7. \( \frac{108}{189} \)
8. \( \frac{75}{500} \)
9. \( \frac{100}{28} \)
10. \( \frac{98}{147} \)
11. \( \frac{1125}{175} \)
12. \( \frac{3087}{8575} \)

In Exercises 13-18, reduce the given expression to lowest terms. State all restrictions.

13. \( \frac{x^2 - 10x + 9}{5x - 5} \)
14. \( \frac{x^2 - 9x + 20}{x^2 - x - 20} \)
15. \( \frac{x^2 - 2x - 35}{x^2 - 7x} \)
16. \( \frac{x^2 - 15x + 54}{x^2 + 7x - 8} \)
17. \( \frac{x^2 + 2x - 63}{x^2 + 13x + 42} \)
18. \( \frac{x^2 + 13x + 42}{9x + 63} \)

In Exercises 19-24, negate any two parts of the fraction, then factor (if necessary) and cancel common factors to reduce the rational expression to lowest terms. State all restrictions.

19. \( \frac{x + 2}{-x - 2} \)
20. \( \frac{4 - x}{x - 4} \)
21. \( \frac{2x - 6}{3 - x} \)
22. \( \frac{3x + 12}{-x - 4} \)
23. \( \frac{3x^2 + 6x}{-x - 2} \)

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24. \( \frac{8x - 2x^2}{x - 4} \)

In Exercises 25-38, reduce each of the given rational expressions to lowest terms. State all restrictions.

25. \( \frac{x^2 - x - 20}{25 - x^2} \)

26. \( \frac{x - x^2}{x^2 - 3x + 2} \)

27. \( \frac{x^2 + 3x - 28}{x^2 + 5x - 36} \)

28. \( \frac{x^2 + 10x + 9}{x^2 + 15x + 54} \)

29. \( \frac{x^2 - x - 56}{8x - x^2} \)

30. \( \frac{x^2 - 7x + 10}{5x - x^2} \)

31. \( \frac{x^2 + 13x + 42}{x^2 - 2x - 63} \)

32. \( \frac{x^2 - 16}{x^2 - x - 12} \)

33. \( \frac{x^2 - 9x + 14}{49 - x^2} \)

34. \( \frac{x^2 + 7x + 12}{9 - x^2} \)

35. \( \frac{x^2 - 3x - 18}{x^2 - 6x + 5} \)

36. \( \frac{x^2 + 5x - 6}{x^2 - 1} \)

37. \( \frac{x^2 - 3x - 10}{-9x - 18} \)

38. \( \frac{x^2 - 6x + 8}{16 - x^2} \)

In Exercises 39-42, reduce each rational function to lowest terms, and then perform each of the following tasks.

i. Load the original rational expression into Y1 and the reduced rational expression (your answer) into Y2 of your graphing calculator.

ii. In TABLE SETUP, set TblStart equal to zero, \( \Delta \text{Tbl} \) equal to 1, then make sure both independent and dependent variables are set to Auto. Select TABLE and scroll with the up- and down-arrows on your calculator until the smallest restriction is in view. Copy both columns of the table onto your homework paper, showing the agreement between Y1 and Y2 and what happens at all restrictions.

39. \( \frac{x^2 - 8x + 7}{x^2 - 11x + 28} \)

40. \( \frac{x^2 - 5x}{x^2 - 9x} \)

41. \( \frac{8x - x^2}{x^2 - x - 56} \)

42. \( \frac{x^2 + 13x + 40}{-2x - 16} \)

Given \( f(x) = 2x + 5 \), simplify each of the expressions in Exercises 43-46. Be sure to reduce your answer to lowest terms and state any restrictions.

43. \( \frac{f(x) - f(3)}{x - 3} \)

44. \( \frac{f(x) - f(6)}{x - 6} \)

45. \( \frac{f(x) - f(a)}{x - a} \)

46. \( \frac{f(a + h) - f(a)}{h} \)
Given \( f(x) = x^2 + 2x \), simplify each of the expressions in Exercises 47-50. Be sure to reduce your answer to lowest terms and state any restrictions.

47. \[ \frac{f(x) - f(1)}{x - 1} \]

48. \[ \frac{f(x) - f(a)}{x - a} \]

49. \[ \frac{f(a + h) - f(a)}{h} \]

50. \[ \frac{f(x + h) - f(x)}{h} \]

**Drill for Skill.** In Exercises 51-54, evaluate the given function at the given expression and simplify your answer.

51. Suppose that \( f \) is the function

\[ f(x) = \frac{x - 6}{8x + 7}. \]

Evaluate \( f(-3x + 2) \) and simplify your answer.

52. Suppose that \( f \) is the function

\[ f(x) = \frac{5x + 3}{7x + 6}. \]

Evaluate \( f(-5x + 1) \) and simplify your answer.

53. Suppose that \( f \) is the function

\[ f(x) = \frac{3x - 6}{4x + 6}. \]

Evaluate \( f(-x-3) \) and simplify your answer.

54. Suppose that \( f \) is the function

\[ f(x) = \frac{4x - 1}{2x - 4}. \]

Evaluate \( f(5x) \) and simplify your answer.
### 7.2 Answers

1. \( \frac{3}{2} \)

3. \( \frac{35}{4} \)

5. \( \frac{35}{9} \)

7. \( \frac{4}{7} \)

9. \( \frac{25}{7} \)

11. \( \frac{45}{7} \)

13. \( \frac{x - 9}{5} \), provided \( x \neq 1 \)

15. \( \frac{x + 5}{x} \), provided \( x \neq 0, 7 \)

17. \( \frac{(x - 7)(x + 9)}{(x + 7)(x + 6)} \), provided \( x \neq -7, -6 \)

19. \( -1 \), provided \( x \neq -2 \)

21. \( -2 \), provided \( x \neq 3 \)

23. \( -3x \), provided \( x \neq -2 \)

25. \( -\frac{x + 4}{x + 5} \), provided \( x \neq -5, 5 \)

27. \( \frac{x + 7}{x + 9} \), provided \( x \neq 4, -9 \)

29. \( -\frac{x + 7}{x} \), provided \( x \neq 0, 8 \)

31. \( \frac{x + 6}{x - 9} \), provided \( x \neq -7, 9 \)

33. \( -\frac{x - 2}{x + 7} \), provided \( x \neq 7, -7 \)

35. \( \frac{(x - 6)(x + 3)}{(x - 1)(x - 5)} \), provided \( x \neq 1, 5 \)

37. \( -\frac{x - 5}{9} \), provided \( x \neq -2 \)

39. \( \frac{x - 1}{x - 4} \), provided \( x \neq 7, 4 \)
41. \(-\frac{x}{x + 7}\), provided \(x \neq -7, 8\)

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<th>Y2</th>
</tr>
</thead>
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<td></td>
</tr>
<tr>
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</tr>
<tr>
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<tr>
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<tr>
<td>9</td>
<td>-0.5625</td>
<td>-0.5625</td>
</tr>
</tbody>
</table>

43. \(\frac{2}{x}\), provided \(x \neq 3\)

45. \(\frac{2}{x}\), provided \(x \neq a\)

47. \(x + 3\), provided \(x \neq 1\)

49. \(2a + h + 2\), provided \(h \neq 0\)

51. \(-\frac{3x + 4}{24x - 23}\)

53. \(-\frac{3x + 15}{4x + 6}\)
7.3 Graphing Rational Functions

We’ve seen that the denominator of a rational function is never allowed to equal zero; division by zero is not defined. So, with rational functions, there are special values of the independent variable that are of particular importance. Now, it comes as no surprise that near values that make the denominator zero, rational functions exhibit special behavior, but here, we will also see that values that make the numerator zero sometimes create additional special behavior in rational functions.

We begin our discussion by focusing on the domain of a rational function.

The Domain of a Rational Function

When presented with a rational function of the form

\[ f(x) = \frac{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n}{b_0 + b_1x + b_2x^2 + \cdots + b_mx^m}, \]  

(1)

the first thing we must do is identify the domain. Equivalently, we must identify the restrictions, values of the independent variable (usually \(x\)) that are not in the domain.

To facilitate the search for restrictions, we should factor the denominator of the rational function (it won’t hurt to factor the numerator at this time as well, as we will soon see). Once the domain is established and the restrictions are identified, here are the pertinent facts.

Behavior of a Rational Function at Its Restrictions. A rational function can only exhibit one of two behaviors at a restriction (a value of the independent variable that is not in the domain of the rational function).

1. The graph of the rational function will have a vertical asymptote at the restricted value.
2. The graph will exhibit a “hole” at the restricted value.

In the next two examples, we will examine each of these behaviors. In this first example, we see a restriction that leads to a vertical asymptote.

Example 2. Sketch the graph of

\[ f(x) = \frac{1}{x + 2}. \]

The first step is to identify the domain. Note that \(x = -2\) makes the denominator of \(f(x) = 1/(x + 2)\) equal to zero. Division by zero is undefined. Hence, \(x = -2\) is not in the domain of \(f\); that is, \(x = -2\) is a restriction. Equivalently, the domain of \(f\) is \(\{x : x \neq -2\}\).

Now that we’ve identified the restriction, we can use the theory of Section 7.1 to shift the graph of \(y = 1/x\) two units to the left to create the graph of \(f(x) = 1/(x + 2)\), as shown in Figure 1.

---

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Version: Fall 2007
The function \( f(x) = 1/(x + 2) \) has a restriction at \( x = -2 \).
The graph of \( f \) has a vertical asymptote with equation \( x = -2 \).

It is important to note that although the restricted value \( x = -2 \) makes the denominator of \( f(x) = 1/(x + 2) \) equal to zero, it does not make the numerator equal to zero. We’ll soon have more to say about this observation.

Let’s look at an example of a rational function that exhibits a “hole” at one of its restricted values.

**Example 3.** Sketch the graph of

\[
f(x) = \frac{x - 2}{x^2 - 4}.
\]

We highlight the first step.

**Factor Numerators and Denominators.** When working with rational functions, the first thing you should always do is factor both numerator and denominator of the rational function.

Following this advice, we factor both numerator and denominator of \( f(x) = (x - 2)/(x^2 - 4) \).

\[
f(x) = \frac{x - 2}{(x - 2)(x + 2)}
\]

It is easier to spot the restrictions when the denominator of a rational function is in factored form. Clearly, \( x = -2 \) and \( x = 2 \) will both make the denominator of
\[ f(x) = \frac{(x - 2)/((x - 2)(x + 2))}{1} \text{ equal to zero. Hence, } x = -2 \text{ and } x = 2 \text{ are restrictions of the rational function } f. \]

Now that the restrictions of the rational function \( f \) are established, we proceed to the second step.

**Reduce to Lowest Terms.** After you establish the restrictions of the rational function, the second thing you should do is reduce the rational function to lowest terms.

Following this advice, we cancel common factors and reduce the rational function \( f(x) = \frac{(x - 2)/((x - 2)(x + 2))}{1} \) to lowest terms, obtaining a new function,

\[ g(x) = \frac{1}{x + 2}. \]

The functions \( f(x) = \frac{(x - 2)/((x - 2)(x + 2))}{1} \) and \( g(x) = \frac{1}{x + 2} \) are not identical functions. They have different domains. The domain of \( f \) is \( D_f = \{x : x \neq -2, 2\} \), but the domain of \( g \) is \( D_g = \{x : x \neq -2\} \). Hence, the only difference between the two functions occurs at \( x = 2 \). The number 2 is in the domain of \( g \), but not in the domain of \( f \).

We know what the graph of the function \( g(x) = \frac{1}{x + 2} \) looks like. We drew this graph in **Example 2** and we picture it anew in **Figure 2**.

**Figure 2.** The graph of \( g(x) = \frac{1}{x + 2} \) exhibits a vertical asymptote at its restriction \( x = -2 \).

The difficulty we now face is the fact that we’ve been asked to draw the graph of \( f \), not the graph of \( g \). However, we know that the functions \( f \) and \( g \) agree at all values of \( x \) except \( x = 2 \). If we remove this value from the graph of \( g \), then we will have the graph of \( f \).
So, what point should we remove from the graph of \( g \)? We should remove the point that has an \( x \)-value equal to 2. Therefore, we evaluate the function \( g(x) = \frac{1}{x + 2} \) at \( x = 2 \) and find

\[
g(2) = \frac{1}{2 + 2} = \frac{1}{4}.
\]

Because \( g(2) = \frac{1}{4} \), we remove the point \((2, 1/4)\) from the graph of \( g \) to produce the graph of \( f \). The result is shown in Figure 3.

![Figure 3](image.png)

**Figure 3.** The graph of \( f(x) = \frac{(x - 2)}{(x - 2)(x + 2)} \) exhibits a vertical asymptote at its restriction \( x = -2 \) and a hole at its second restriction \( x = 2 \).

We pause to make an important observation. In Example 3, we started with the function

\[
f(x) = \frac{x - 2}{(x - 2)(x + 2)},
\]

which had restrictions at \( x = 2 \) and \( x = -2 \). After reducing, the function

\[
g(x) = \frac{1}{x + 2}
\]

no longer had a restriction at \( x = 2 \). The function \( g \) had a single restriction at \( x = -2 \). The result, as seen in Figure 3, was a vertical asymptote at the remaining restriction, and a hole at the restriction that “went away” due to cancellation. This leads us to the following procedure.
Asymptote or Hole? To determine whether the graph of a rational function has a vertical asymptote or a hole at a restriction, proceed as follows:

1. Factor numerator and denominator of the original rational function \( f \). Identify the restrictions of \( f \).
2. Reduce the rational function to lowest terms, naming the new function \( g \). Identify the restrictions of the function \( g \).
3. Those restrictions of \( f \) that remain restrictions of the function \( g \) will introduce vertical asymptotes into the graph of \( f \).
4. Those restrictions of \( f \) that are no longer restrictions of the function \( g \) will introduce “holes” into the graph of \( f \). To determine the coordinates of the holes, substitute each restriction of \( f \) that is not a restriction of \( g \) into the function \( g \) to determine the \( y \)-value of the hole.

We now turn our attention to the zeros of a rational function.

The Zeros of a Rational Function

We’ve seen that division by zero is undefined. That is, if we have a fraction \( N/D \), then \( D \) (the denominator) must not equal zero. Thus, 5/0, −15/0, and 0/0 are all undefined. On the other hand, in the fraction \( N/D \), if \( N = 0 \) and \( D \neq 0 \), then the fraction is equal to zero. For example, 0/5, 0/(-15), and 0/\( \pi \) are all equal to zero.

Therefore, when working with an arbitrary rational function, such as

\[
f(x) = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m},
\]

(4)

whatever value of \( x \) that will make the numerator zero without simultaneously making the denominator equal to zero will be a zero of the rational function \( f \).

This discussion leads to the following procedure for identifying the zeros of a rational function.

Finding Zeros of Rational Functions. To determine the zeros of a rational function, proceed as follows.

1. Factor both numerator and denominator of the rational function \( f \).
2. Identify the restrictions of the rational function \( f \).
3. Identify the values of the independent variable (usually \( x \)) that make the numerator equal to zero.
4. The zeros of the rational function \( f \) will be those values of \( x \) that make the numerator zero but are not restrictions of the rational function \( f \).

Let’s look at an example.

Example 5. Find the zeros of the rational function defined by
\[ f(x) = \frac{x^2 + 3x + 2}{x^2 - 2x - 3}. \]  

(6)

Factor numerator and denominator of the rational function \( f \).

\[ f(x) = \frac{(x + 1)(x + 2)}{(x + 1)(x - 3)} \]

The values \( x = -1 \) and \( x = 3 \) make the denominator equal to zero and are restrictions.

Next, note that \( x = -1 \) and \( x = -2 \) both make the numerator equal to zero. However, \( x = -1 \) is also a restriction of the rational function \( f \), so it will not be a zero of \( f \). On the other hand, the value \( x = -2 \) is not a restriction and will be a zero of \( f \).

Although we’ve correctly identified the zeros of \( f \), it’s instructive to check the values of \( x \) that make the numerator of \( f \) equal to zero. If we substitute \( x = -1 \) into original function defined by equation (6), we find that

\[ f(-1) = \frac{(-1)^2 + 3(-1) + 2}{(-1)^2 - 2(-1) - 3} = \frac{0}{0} \]

is undefined. Hence, \( x = -1 \) is not a zero of the rational function \( f \). The difficulty in this case is that \( x = -1 \) also makes the denominator equal to zero.

On the other hand, when we substitute \( x = -2 \) in the function defined by equation (6),

\[ f(-2) = \frac{(-2)^2 + 3(-2) + 2}{(-2)^2 - 2(-2) - 3} = \frac{0}{5} = 0. \]

In this case, \( x = -2 \) makes the numerator equal to zero without making the denominator equal to zero. Hence, \( x = -2 \) is a zero of the rational function \( f \).

It’s important to note that you must work with the original rational function, and not its reduced form, when identifying the zeros of the rational function.

\textbf{Example 7.} \textbf{Identify the zeros of the rational function}

\[ f(x) = \frac{x^2 - 6x + 9}{x^2 - 9}. \]

Factor both numerator and denominator.

\[ f(x) = \frac{(x - 3)^2}{(x + 3)(x - 3)} \]

Note that \( x = -3 \) and \( x = 3 \) are restrictions. Further, the only value of \( x \) that will make the numerator equal to zero is \( x = 3 \). However, this is also a restriction. Hence, the function \( f \) has no zeros.

The point to make here is what would happen if you work with the reduced form of the rational function in attempting to find its zeros. Cancelling like factors leads to a new function,
Note that $g$ has only one restriction, $x = -3$. Further, $x = 3$ makes the numerator of $g$ equal to zero and is not a restriction. Hence, $x = 3$ is a zero of the function $g$, but it is not a zero of the function $f$.

This example demonstrates that we must identify the zeros of the rational function before we cancel common factors.

\section*{Drawing the Graph of a Rational Function}

In this section we will use the zeros and asymptotes of the rational function to help draw the graph of a rational function. We will also investigate the end-behavior of rational functions. Let’s begin with an example.

\textbf{Example 8. Sketch the graph of the rational function}

$$f(x) = \frac{x + 2}{x - 3}.$$  \hfill (9)

First, note that both numerator and denominator are already factored. The function has one restriction, $x = 3$. Next, note that $x = -2$ makes the numerator of equation (9) zero and is not a restriction. Hence, $x = -2$ is a zero of the function. Recall that a function is zero where its graph crosses the horizontal axis. Hence, the graph of $f$ will cross the $x$-axis at $(-2, 0)$, as shown in Figure 4.

Note that the rational function (9) is already reduced to lowest terms. Hence, the restriction at $x = 3$ will place a vertical asymptote at $x = 3$, which is also shown in Figure 4.

At this point, we know two things:
1. The graph will cross the $x$-axis at $(-2, 0)$.

2. On each side of the vertical asymptote at $x = 3$, one of two things can happen. Either the graph will rise to positive infinity or the graph will fall to negative infinity.

To discover the behavior near the vertical asymptote, let’s plot one point on each side of the vertical asymptote, as shown in Figure 5.

Consider the right side of the vertical asymptote and the plotted point $(4, 6)$ through which our graph must pass. As the graph approaches the vertical asymptote at $x = 3$, only one of two things can happen. Either the graph rises to positive infinity or the graph falls to negative infinity. However, in order for the latter to happen, the graph must first pass through the point $(4, 6)$, then cross the $x$-axis between $x = 3$ and $x = 4$ on its descent to minus infinity. But we already know that the only $x$-intercept is at the point $(2, 0)$, so this cannot happen. Hence, on the right, the graph must pass through the point $(4, 6)$, then rise to positive infinity, as shown in Figure 6.
A similar argument holds on the left of the vertical asymptote at \( x = 3 \). The graph cannot pass through the point \((2, -4)\) and rise to positive infinity as it approaches the vertical asymptote, because to do so would require that it cross the \( x \)-axis between \( x = 2 \) and \( x = 3 \). However, there is no \( x \)-intercept in this region available for this purpose. Hence, on the left, the graph must pass through the point \((2, -4)\) and fall to negative infinity as it approaches the vertical asymptote at \( x = 3 \). This behavior is shown in Figure 6.

Finally, what about the end-behavior of the rational function? What happens to the graph of the rational function as \( x \) increases without bound? What happens when \( x \) decreases without bound? One simple way to answer these questions is to use a table to investigate the behavior numerically. The graphing calculator facilitates this task.

First, enter your function as shown in Figure 7(a), then press 2nd TBLSET to open the window shown in Figure 7(b). For what we are about to do, all of the settings in this window are irrelevant, save one. Make sure you use the arrow keys to highlight ASK for the Indpnt (independent) variable and press ENTER to select this option. Finally, select 2nd TABLE, then enter the \( x \)-values 10, 100, 1000, and 10000, pressing ENTER after each one.

![Figure 7](image-url)

**Figure 7.** Using the table feature of the graphing calculator to investigate the end-behavior as \( x \) approaches positive infinity.

Note the resulting \( y \)-values in the second column of the table (the Y1 column) in Figure 7(c). As \( x \) is increasing without bound, the \( y \)-values are greater than 1, yet appear to be approaching the number 1. Therefore, as our graph moves to the extreme right, it must approach the horizontal asymptote at \( y = 1 \), as shown in Figure 9.

A similar effort predicts the end-behavior as \( x \) decreases without bound, as shown in the sequence of pictures in Figure 8. As \( x \) decreases without bound, the \( y \)-values are less than 1, but again approach the number 1, as shown in Figure 8(c).
Figure 8. Using the table feature of the graphing calculator to investigate the end-behavior as $x$ approaches negative infinity.

The evidence in Figure 8(c) indicates that as our graph moves to the extreme left, it must approach the horizontal asymptote at $y = 1$, as shown in Figure 9.

![Figure 9](image)

Figure 9. The graph approaches the horizontal asymptote $y = 1$ at the extreme right- and left-ends.

What kind of job will the graphing calculator do with the graph of this rational function? In Figure 10(a), we enter the function, adjust the window parameters as shown in Figure 10(b), then push the GRAPH button to produce the result in Figure 10(c).

![Figure 10](image)

Figure 10. Drawing the graph of the rational function with the graphing calculator.

As was discussed in the first section, the graphing calculator manages the graphs of “continuous” functions extremely well, but has difficulty drawing graphs with discontinuities. In the case of the present rational function, the graph “jumps” from negative...
infinity to positive infinity across the vertical asymptote $x = 3$. The calculator knows only one thing: plot a point, then connect it to the previously plotted point with a line segment. Consequently, it does what it is told, and “connects” infinities when it shouldn’t.

However, if we have prepared in advance, identifying zeros and vertical asymptotes, then we can interpret what we see on the screen in Figure 10(c), and use that information to produce the correct graph that is shown in Figure 9. We can even add the horizontal asymptote to our graph, as shown in the sequence in Figure 11.

![Figure 11](image_url)

**Figure 11.** Adding a suspected horizontal asymptote.

This is an appropriate point to pause and summarize the steps required to draw the graph of a rational function.
Procedure for Graphing Rational Functions. Consider the rational function
\[ f(x) = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m}. \]

To draw the graph of this rational function, proceed as follows:

1. Factor the numerator and denominator of the rational function \( f \).
2. Identify the domain of the rational function \( f \) by listing each restriction, values of the independent variable (usually \( x \)) that make the denominator equal to zero.
3. Identify the values of the independent variable that make the numerator of \( f \) equal to zero and are not restrictions. These are the zeros of \( f \) and they provide the \( x \)-coordinates of the \( x \)-intercepts of the graph of the rational function. Plot these intercepts on a coordinate system and label them with their coordinates.
4. Cancel common factors to reduce the rational function to lowest terms. The restrictions of \( f \) that remain restrictions of this reduced form will place vertical asymptotes in the graph of \( f \). Draw the vertical asymptotes on your coordinate system as dashed lines and label them with their equations.
   - The restrictions of \( f \) that are not restrictions of the reduced form will place “holes” in the graph of \( f \). We’ll deal with the holes in step 8 of this procedure.
5. To determine the behavior near each vertical asymptote, calculate and plot one point on each side of each vertical asymptote.
6. To determine the end-behavior of the given rational function, use the table capability of your calculator to determine the limit of the function as \( x \) approaches positive and/or negative infinity (as we did in the sequences shown in Figure 7 and Figure 8). This determines the horizontal asymptote. Sketch the horizontal asymptote as a dashed line on your coordinate system and label it with its equation.
7. Draw the graph of the rational function.
8. If you determined that a restriction was a “hole,” use the restriction and the reduced form of the rational function to determine the \( y \)-value of the “hole.” Draw an open circle at this position to represent the “hole” and label the “hole” with its coordinates.
9. Finally, use your calculator to check the validity of your result.

Let’s look at another example.

Example 10. Sketch the graph of the rational function
\[ f(x) = \frac{x - 2}{x^2 - 3x - 4}. \] (11)

We will follow the outline presented in the Procedure for Graphing Rational Functions.

Step 1: First, factor both numerator and denominator.
Section 7.3 Graphing Rational Functions

\[ f(x) = \frac{x - 2}{(x + 1)(x - 4)} \]  \hspace{1cm} (12)

**Step 2:** Thus, \( f \) has two restrictions, \( x = -1 \) and \( x = 4 \). That is, the domain of \( f \) is \( D_f = \{ s : x \neq -1, 4 \} \).

**Step 3:** The numerator of equation (12) is zero at \( x = 2 \) and this value is not a restriction. Thus, 2 is a zero of \( f \) and \((2, 0)\) is an \( x \)-intercept of the graph of \( f \), as shown in Figure 12.

**Step 4:** Note that the rational function is already reduced to lowest terms (if it weren’t, we’d reduce at this point). Note that the restrictions \( x = -1 \) and \( x = 4 \) are still restrictions of the reduced form. Hence, these are the locations and equations of the vertical asymptotes, which are also shown in Figure 12.

![Figure 12. Plot the x-intercepts and draw the vertical asymptotes.](image)

All of the restrictions of the original function remain restrictions of the reduced form. Therefore, there will be no “holes” in the graph of \( f \).

**Step 5:** Plot points to the immediate right and left of each asymptote, as shown in Figure 13. These additional points completely determine the behavior of the graph near each vertical asymptote. For example, consider the point \((5, 1/2)\) to the immediate right of the vertical asymptote \( x = 4 \) in Figure 13. Because there is no \( x \)-intercept between \( x = 4 \) and \( x = 5 \), and the graph is already above the \( x \)-axis at the point \((5, 1/2)\), the graph is forced to increase to positive infinity as it approaches the vertical asymptote \( x = 4 \). Similar comments are in order for the behavior on each side of each vertical asymptote.

**Step 6:** Use the table utility on your calculator to determine the end-behavior of the rational function as \( x \) decreases and/or increases without bound. To determine the end-behavior as \( x \) goes to infinity (increases without bound), enter the equation in your calculator, as shown in Figure 14(a). Select 2nd TBLSET and highlight ASK for the independent variable. Select 2nd TABLE, then enter 10, 100, 1000, and 10000, as shown in Figure 14(c).
Chapter 7  Rational Functions

Figure 13. Additional points help determine the behavior near the vertical asymptote.

Figure 14. Examining end-behavior as $x$ approaches positive infinity.

If you examine the $y$-values in Figure 14(c), you see that they are heading towards zero ($1e-4$ means $1 \times 10^{-4}$, which equals 0.0001). This implies that the line $y = 0$ (the $x$-axis) is acting as a horizontal asymptote.

You can also determine the end-behavior as $x$ approaches negative infinity (decreases without bound), as shown in the sequence in Figure 15. The result in Figure 15(c) provides clear evidence that the $y$-values approach zero as $x$ goes to negative infinity. Again, this makes $y = 0$ a horizontal asymptote.

Figure 15. Examining end-behavior as $x$ approaches negative infinity.

Add the horizontal asymptote $y = 0$ to the image in Figure 13.

Step 7: We can use all the information gathered to date to draw the image shown in Figure 16.
Step 8: As stated above, there are no “holes” in the graph of $f$.

Step 9: Use your graphing calculator to check the validity of your result. Note how the graphing calculator handles the graph of this rational function in the sequence in Figure 17. The image in Figure 17(c) is nowhere near the quality of the image we have in Figure 16, but there is enough there to intuit the actual graph if you prepare properly in advance (zeros, vertical asymptotes, end-behavior analysis, etc.).

Figure 17. The user of the graphing calculator must decipher the image in the calculator’s view screen.
7.3 Exercises

For rational functions Exercises 1-20, follow the Procedure for Graphing Rational Functions in the narrative, performing each of the following tasks.

For rational functions Exercises 1-20, perform each of the following tasks.

i. Set up a coordinate system on graph paper. Label and scale each axis. Remember to draw all lines with a ruler.

ii. Perform each of the nine steps listed in the Procedure for Graphing Rational Functions in the narrative.

1. \( f(x) = \frac{x - 3}{x + 2} \)
2. \( f(x) = \frac{x + 2}{x - 4} \)
3. \( f(x) = \frac{5 - x}{x + 1} \)
4. \( f(x) = \frac{x + 2}{4 - x} \)
5. \( f(x) = \frac{2x - 5}{x + 1} \)
6. \( f(x) = \frac{2x + 5}{3 - x} \)
7. \( f(x) = \frac{x + 2}{x^2 - 2x - 3} \)
8. \( f(x) = \frac{x - 3}{x^2 - 3x - 4} \)
9. \( f(x) = \frac{x + 1}{x^2 + x - 2} \)
10. \( f(x) = \frac{x - 1}{x^2 - x - 2} \)
11. \( f(x) = \frac{x^2 - 2x}{x^2 + x - 2} \)
12. \( f(x) = \frac{x^2 - 2x}{x^2 - 2x - 8} \)
13. \( f(x) = \frac{2x^2 - 2x - 4}{x^2 - x - 12} \)
14. \( f(x) = \frac{8x - 2x^2}{x^2 - x - 6} \)
15. \( f(x) = \frac{x - 3}{x^2 - 5x + 6} \)
16. \( f(x) = \frac{2x - 4}{x^2 - x - 2} \)
17. \( f(x) = \frac{2x^2 - x - 6}{x^2 - 2x} \)
18. \( f(x) = \frac{2x^2 - x - 6}{x^2 - 2x} \)
19. \( f(x) = \frac{4 + 2x - 2x^2}{x^2 + 4x + 3} \)
20. \( f(x) = \frac{3x^2 - 6x - 9}{1 - x^2} \)

In Exercises 21-28, find the coordinates of the x-intercept(s) of the graph of the given rational function.

21. \( f(x) = \frac{81 - x^2}{x^2 + 10x + 9} \)
22. \( f(x) = \frac{x - x^2}{x^2 + 5x - 6} \)
23. \( f(x) = \frac{x^2 - x - 12}{x^2 + 2x - 3} \)
24. \( f(x) = \frac{x^2 - 81}{x^2 - 4x - 45} \)
25. \( f(x) = \frac{6x - 18}{x^2 - 7x + 12} \)
26. \( f(x) = \frac{4x + 36}{x^2 + 15x + 54} \)
27. \( f(x) = \frac{x^2 - 9x + 14}{x^2 - 2x} \)
28. \( f(x) = \frac{x^2 - 5x - 36}{x^2 - 9x + 20} \)

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In Exercises 29-36, find the equations of all vertical asymptotes.

29. \( f(x) = \frac{x^2 - 7x}{x^2 - 2x} \)

30. \( f(x) = \frac{x^2 + 4x - 45}{3x + 27} \)

31. \( f(x) = \frac{x^2 - 6x + 8}{x^2 - 16} \)

32. \( f(x) = \frac{x^2 - 11x + 18}{2x - x^2} \)

33. \( f(x) = \frac{x^2 + x - 12}{-4x + 12} \)

34. \( f(x) = \frac{x^2 - 3x - 54}{9x - x^2} \)

35. \( f(x) = \frac{16 - x^2}{x^2 + 7x + 12} \)

36. \( f(x) = \frac{x^2 - 11x + 30}{-8x + 48} \)

In Exercises 37-42, use a graphing calculator to determine the behavior of the given rational function as \( x \) approaches both positive and negative infinity by performing the following tasks:

i. Load the rational function into the \( Y= \) menu of your calculator.

ii. Use the TABLE feature of your calculator to determine the value of \( f(x) \) for \( x = 10, 100, 1000, \) and 10000. Record these results on your homework in table form.

iii. Use the TABLE feature of your calculator to determine the value of \( f(x) \) for \( x = -10, -100, -1000, \) and \(-10000\). Record these results on your homework in table form.

iv. Use the results of your tabular exploration to determine the equation of

the horizontal asymptote.

37. \( f(x) = \frac{2x + 3}{x - 8} \)

38. \( f(x) = \frac{4 - 3x}{x + 2} \)

39. \( f(x) = \frac{4 - x^2}{x^2 + 4x + 3} \)

40. \( f(x) = \frac{10 - 2x^2}{x^2 - 4} \)

41. \( f(x) = \frac{x^2 - 2x - 3}{2x^2 - 3x - 2} \)

42. \( f(x) = \frac{2x^2 - 3x - 5}{x^2 - x - 6} \)

In Exercises 43-48, use a purely analytical method to determine the domain of the given rational function. Describe the domain using set-builder notation.

43. \( f(x) = \frac{x^2 - 5x - 6}{-9x - 9} \)

44. \( f(x) = \frac{x^2 + 4x + 3}{x^2 - 5x - 6} \)

45. \( f(x) = \frac{x^2 + 5x - 24}{x^2 - 3x} \)

46. \( f(x) = \frac{x^2 - 3x - 4}{x^2 - 5x - 6} \)

47. \( f(x) = \frac{x^2 - 4x + 3}{x - x^2} \)

48. \( f(x) = \frac{x^2 - 4}{x^2 - 9x + 14} \)
7.3 Answers

1. 

7. 

3. 

9. 

5. 

11. 

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13. $\frac{y}{x} = \frac{-3}{4}$

15. $\frac{y}{x} = 0$

17. $\frac{y}{x} = \frac{-3}{2}$

19. $\frac{y}{x} = 2$

21. $(9, 0)$

23. $(4, 0)$

25. no $x$-intercepts

27. $(7, 0)$

29. $x = 2$

31. $x = -4$

33. no vertical asymptotes

35. $x = -3$

37. Horizontal asymptote at $y = 2.$

<table>
<thead>
<tr>
<th>X</th>
<th>Y1</th>
<th>X</th>
<th>Y1</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>11.5</td>
<td>-10</td>
<td>0.944444</td>
</tr>
<tr>
<td>100</td>
<td>2.20652</td>
<td>-100</td>
<td>1.82407</td>
</tr>
<tr>
<td>1000</td>
<td>2.01915</td>
<td>-1000</td>
<td>1.98115</td>
</tr>
<tr>
<td>10000</td>
<td>2.0019</td>
<td>-10000</td>
<td>1.998</td>
</tr>
</tbody>
</table>

39. Horizontal asymptote at $y = -1.$

<table>
<thead>
<tr>
<th>X</th>
<th>Y1</th>
<th>X</th>
<th>Y1</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>-0.671329</td>
<td>-10</td>
<td>-1.52381</td>
</tr>
<tr>
<td>100</td>
<td>-0.960877</td>
<td>-100</td>
<td>-1.04092</td>
</tr>
<tr>
<td>1000</td>
<td>-0.996009</td>
<td>-1000</td>
<td>-1.00401</td>
</tr>
<tr>
<td>10000</td>
<td>-0.9996</td>
<td>-10000</td>
<td>-1</td>
</tr>
</tbody>
</table>
41. Horizontal asymptote at $y = 1/2$.

<table>
<thead>
<tr>
<th>X</th>
<th>Y1</th>
<th>X</th>
<th>Y1</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.458333</td>
<td>-10</td>
<td>0.513158</td>
</tr>
<tr>
<td>100</td>
<td>0.49736</td>
<td>-100</td>
<td>0.502365</td>
</tr>
<tr>
<td>1000</td>
<td>0.499749</td>
<td>-1000</td>
<td>0.500249</td>
</tr>
<tr>
<td>10000</td>
<td>0.49997</td>
<td>-10000</td>
<td>0.50002</td>
</tr>
</tbody>
</table>

43. Domain = \( \{ x : x \neq -1 \} \)

45. Domain = \( \{ x : x \neq 3, 0 \} \)

47. Domain = \( \{ x : x \neq 0, 1 \} \)
7.4 Products and Quotients of Rational Functions

In this section we deal with products and quotients of rational expressions. Before we begin, we’ll need to establish some fundamental definitions and technique. We begin with the definition of the product of two rational numbers.

**Definition 1.** Let \( \frac{a}{b} \) and \( \frac{c}{d} \) be rational numbers. The product of these rational numbers is defined by

\[
\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d},
\]

or more compactly,

\[
\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.
\]

The definition simply states that you should multiply the numerators of each rational number to obtain the numerator of the product, and you also multiply the denominators of each rational number to obtain the denominator of the product. For example,

\[
\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}.
\]

Of course, you should also check to make sure your final answer is reduced to lowest terms.

Let’s look at an example.

**Example 3.** Simplify the product of rational numbers

\[
\frac{6}{231} \cdot \frac{35}{10}.
\]

First, multiply numerators and denominators together as follows.

\[
\frac{6}{231} \cdot \frac{35}{10} = \frac{6 \cdot 35}{231 \cdot 10} = \frac{210}{2310}.
\]

However, the answer is not reduced to lowest terms. We can express the numerator as a product of primes.

\[
210 = 21 \cdot 10 = 3 \cdot 7 \cdot 2 \cdot 5 = 2 \cdot 3 \cdot 5 \cdot 7
\]

It’s not necessary to arrange the factors in ascending order, but every little bit helps. The denominator can also be expressed as a product of primes.

\[
2310 = 10 \cdot 231 = 2 \cdot 5 \cdot 7 \cdot 33 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11
\]

We can now cancel common factors.

\[
\frac{210}{2310} = \frac{2 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11} = \frac{2 \cdot \cancel{3} \cdot \cancel{5} \cdot \cancel{7} \cdot \cancel{11}}{\cancel{2} \cdot \cancel{3} \cdot \cancel{5} \cdot \cancel{7} \cdot 11} = \frac{1}{11}
\]

---

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However, this approach is not the most efficient way to proceed, as multiplying numerators and denominators allows the products to grow to larger numbers, as in $210/2310$. It is then a little bit harder to prime factor the larger numbers.

A better approach is to factor the smaller numerators and denominators immediately, as follows.

\[
\frac{6 \cdot 35}{231 \cdot 10} = \frac{2 \cdot 3 \cdot 5 \cdot 7}{3 \cdot 7 \cdot 11 \cdot 2 \cdot 5}
\]

We could now multiply numerators and denominators, then cancel common factors, which would match identically the last computation in equation (5).

However, we can also employ the following cancellation rule.

**Cancellation Rule.** When working with the product of two or more rational expressions, factor all numerators and denominators, then cancel. The cancellation rule is simple: cancel a factor “on the top” for an identical factor “on the bottom.” Speaking more technically, cancel any factor in any numerator for an identical factor in any denominator.

Thus, we can finish our computation by canceling common factors, canceling “something on the top for something on the bottom.”

\[
\frac{6 \cdot 35}{231 \cdot 10} = \frac{2 \cdot 3 \cdot 5 \cdot 7}{3 \cdot 7 \cdot 11 \cdot 2 \cdot 5} = \frac{2 \cdot 3 \cdot 7 \cdot 11}{2 \cdot 3 \cdot 7 \cdot 11} = \frac{1}{1}
\]

Note that we canceled a 2, 3, 5, and a 7 “on the top” for a 2, 3, 5, and 7 “on the bottom.”

Thus, we have two choices when multiplying rational expressions:

- Multiply numerators and denominators, factor, then cancel.
- Factor numerators and denominators, cancel, then multiply numerators and denominators.

It is the latter approach that we will use in this section. Let’s look at another example.

---

Students will sometimes use the phrase “cross-cancel” when working with the product of rational expressions. Unfortunately, this term implies that cancellation can occur only in a diagonal direction, which is far from the truth. We like to tell our students that there is no such term as “cross-cancel.” There is only “cancel,” and the rule is: cancel something on the top for something on the bottom, which is vernacular for “cancel a factor from any numerator and the identical factor from any denominator.”
Example 6. Simplify the expression
\[
\frac{x^2 - x - 6}{x^2 + 2x - 15} \cdot \frac{x^2 - x - 30}{x^2 - 2x - 8} \tag{7}
\]
State restrictions.

Use the ac-test to factor each numerator and denominator. Then cancel as shown.
\[
\frac{x^2 - x - 6}{x^2 + 2x - 15} \cdot \frac{x^2 - x - 30}{x^2 - 2x - 8} = \frac{(x + 2)(x - 3)}{(x - 3)(x + 5)} \cdot \frac{(x + 5)(x - 6)}{(x + 2)(x - 4)}
\]
\[
= \frac{x - 6}{x - 4}
\]
The first fraction’s denominator has factors \(x - 3\) and \(x + 5\). Hence, \(x = 3\) or \(x = -5\) will make this denominator zero. Therefore, the 3 and -5 are restrictions.

The second fraction’s denominator has factors \(x + 2\) and \(x - 4\). Hence, \(x = -2\) or \(x = 4\) will make this denominator zero. Therefore, -2 and 4 are restrictions.

Therefore, for all values of \(x\), except the restrictions -5, -2, 3, and 4, the left side of
\[
\frac{x^2 - x - 6}{x^2 + 2x - 15} \cdot \frac{x^2 - x - 30}{x^2 - 2x - 8} = \frac{x - 6}{x - 4} \tag{8}
\]
is identical to its right side.

It’s possible to use your graphing calculator to check your results. First, load the left- and right-hand sides of equation (8) into the calculator’s into Y1 and Y2 in your graphing calculator’s Y= menu, as shown in Figure 1(a). Press 2nd TBLSET and set TblStart = -6 and ΔTbl = 1, as shown in Figure 1(b). Make sure that AUTO is highlighted and selected with the ENTER key on both the independent and dependent variables. Press 2nd TABLE to produce the tabular display in Figure 1(c).

Figure 1. Using the table features of the graphing calculator to check the result in equation (8).

Remember that the left- and right-hand sides of equation (8) are loaded in Y1 and Y2, respectively.
• In Figure 1(c), note the ERR (error) message at the restricted values of \(x = -5\) and \(x = -2\). However, other than at these two restrictions, the functions \(Y_1\) and \(Y_2\) agree at all other values of \(x\) in Figure 1(c).

• Use the down arrow to scroll down in the table to produce the tabular results shown in Figure 1(d). Note the ERR (error) message at the restricted values of \(x = 3\) and \(x = 4\). However, other than at these two restrictions, the functions \(Y_1\) and \(Y_2\) agree at all other values of \(x\) in Figure 1(d).

• If you scroll up or down in the table, you’ll find that the functions \(Y_1\) and \(Y_2\) agree at all values of \(x\) other than the restricted values \(-5, -2, 3,\) and \(4\).

Let’s look at another example.

**Example 9.** Simplify

\[
\frac{9 - x^2}{x^2 + 3x} \cdot \frac{6x - 2x^2}{x^2 - 6x + 9}
\]

(10)

State any restrictions.

The first numerator can be factored using the difference of two squares pattern.

\[9 - x^2 = (3 + x)(3 - x).\]

The second denominator is a perfect square trinomial and can be factored as the square of a binomial.

\[x^2 - 6x + 9 = (x - 3)^2\]

You will want to remove the greatest common factor from the first denominator and second numerator.

\[x^2 + 3x = x(x + 3)\quad \text{and}\quad 6x - 2x^2 = 2x(3 - x)\]

Thus,

\[
\frac{9 - x^2}{x^2 + 3x} \cdot \frac{6x - 2x^2}{x^2 - 6x + 9} = \frac{(3 + x)(3 - x)}{x(x + 3)} \cdot \frac{2x(3 - x)}{(x - 3)^2}.
\]

We’ll need to execute a sign change or two to create common factors in the numerators and denominators. So, in both the first and second numerator, factor a \(-1\) from the factor \(3 - x\) to obtain \(3 - x = -1(x - 3)\). Because the order of factors in a product doesn’t matter, we’ll slide the \(-1\) to the front in each case.

\[
\frac{9 - x^2}{x^2 + 3x} \cdot \frac{6x - 2x^2}{x^2 - 6x + 9} = \frac{-(3 + x)(x - 3)}{x(x + 3)} \cdot \frac{-2x(x - 3)}{(x - 3)^2}.
\]

We can now cancel common factors.

Version: Fall 2007
\[
\frac{9 - x^2}{x^2 + 3x} \cdot \frac{6x - 2x^2}{x^2 - 6x + 9} = \frac{-(3 + x)(x - 3)}{x(x + 3)} \cdot \frac{-2x(x - 3)}{(x - 3)^2} \\
= \frac{-(3 + x)(x - 3)}{x(x + 3)} \cdot \frac{-2x(x - 3)}{(x - 3)^2} \\
= 2
\]

A few things to notice:

- The factors \(3 + x\) and \(x + 3\) are identical, so they may be cancelled, one on the top for one on the bottom.
- Two factors of \(x - 3\) on the top are cancelled for \((x - 3)^2\) (which is equivalent to \(x - 3)(x - 3)\) on the bottom.
- An \(x\) on top cancels an \(x\) on the bottom.
- We’re left with two minus signs (two \(-1\)'s) and a 2. So the solution is a positive 2.

Finally, the first denominator has factors \(x\) and \(x + 3\), so \(x = 0\) and \(x = -3\) are restrictions (they make this denominator equal to zero). The second denominator has two factors of \(x - 3\), so \(x = 3\) is an additional restriction.

Hence, for all values of \(x\), except the restricted values \(-3\), 0, and 3, the left-hand side of

\[
\frac{9 - x^2}{x^2 + 3x} \cdot \frac{6x - 2x^2}{x^2 - 6x + 9} = 2
\]

is identical to the right-hand side. Again, this claim is easily tested on the graphing calculator which is evidenced in the sequence of screen captures in Figure 2.

![Figure 2](attachment:image.png)

**Figure 2.** Using the table features of the graphing calculator to check the result in equation (11).

An alternate approach to the problem in equation (10) is to note differing orders in the numerators and denominators (descending, ascending powers of \(x\)) and anticipate the need for a sign change. That is, make the sign change before you factor.

For example, negate (multiply by \(-1\)) both numerator and fraction bar of the first fraction to obtain

\[
\frac{9 - x^2}{x^2 + 3x} = -\frac{x^2 - 9}{x^2 + 3x}
\]

According to our sign change rule, negating any two parts of a fraction leaves the fraction unchanged.
If we perform a similar sign change on the second fraction (negate numerator and fraction bar), then we can factor and cancel common factors.

\[
\frac{9 - x^2}{x^2 + 3x} \cdot \frac{6x - 2x^2}{x^2 - 6x + 9} = \frac{x^2 - 9}{x^2 + 3x} \cdot \frac{2x^2 - 6x}{x^2 - 6x + 9}
\]

\[
= \frac{(x + 3)(x - 3)}{x(x + 3)} \cdot \frac{2x(x - 3)}{(x - 3)^2}
\]

\[
= \frac{2}{x(x - 3)}
\]

\[
= 2
\]

### Division of Rational Expressions

A simple definition will change a problem involving division of two rational expressions into one involving multiplication of two rational expressions. Then there’s nothing left to explain, for we already know how to multiply two rational expressions.

So, let’s motivate our definition of division. Suppose we ask the question, how many halves are in a whole? The answer is easy, as two halves make a whole. Thus, when we divide 1 by 1/2, we should get 2. There are two halves in one whole.

Let’s raise the stakes a bit and ask how many halves are in six? To make the problem more precise, imagine you’ve ordered 6 pizzas and you cut each in half. How many halves do you have? Again, this is easy when you think about the problem in this manner, the answer is 12. Thus,

\[
6 \div \frac{1}{2}
\]

(how many halves are in six) is identical to

\[
6 \cdot 2,
\]

which, of course, is 12. Hopefully, thanks to this opening motivation, the following definition will not seem too strange.

**Definition 12.** To perform the division

\[
\frac{a}{b} \div \frac{c}{d}
\]

invert the second fraction and multiply, as in

\[
\frac{a}{b} \cdot \frac{d}{c}
\]

Thus, if we want to know how many halves are in 12, we change the division into multiplication (“invert and multiply”).
12 ÷ \frac{1}{2} = 12 \cdot 2 = 24

This makes sense, as there are 24 “halves” in 12. Let’s look at a harder example.

Example 13. Simplify

\[ \frac{33}{15} ÷ \frac{14}{10} = \frac{33}{15} \cdot \frac{10}{14} \]

Invert the second fraction and multiply. After that, all we need to do is factor numerators and denominators, then cancel common factors.

\[ \frac{33}{15} ÷ \frac{14}{10} = \frac{33 \cdot 10}{15 \cdot 14} = \frac{3 \cdot 11 \cdot 2 \cdot 5}{3 \cdot 5 \cdot 2 \cdot 7} = \frac{11}{7} \]

An interesting way to check this result on your calculator is shown in the sequence of screens in Figure 3.

![Figure 3. Using the calculator to check division of fractions.](a) (b) (c)

After entering the original problem in your calculator, press ENTER, then press the MATH button, then select 1:► Frac from the menu and press ENTER. The result is shown in Figure 3(c), which agrees with our calculation above.

Let’s look at another example.

Example 15. Simplify

\[ \frac{9 + 3x - 2x^2}{x^2 - 16} ÷ \frac{4x^3 - 9x}{2x^2 + 5x - 12} \]

State the restrictions.

Note the order of the first numerator differs from the other numerators and denominators, so we “anticipate” the need for a sign change, negating the numerator and fraction bar of the first fraction. We also invert the second fraction and change the division to multiplication (“invert and multiply”).

\[ \frac{-2x^2 - 3x - 9}{x^2 - 16} \cdot \frac{2x^2 + 5x - 12}{4x^3 - 9x} \]

The numerator in the first fraction in equation (17) is a quadratic trinomial, with \( ac = (2)(-9) = -18 \). The integer pair 3 and -6 has product -18 and sum -3. Hence,
\[ 2x^2 - 3x - 9 = 2x^2 + 3x - 6x - 9 = x(2x + 3) - 3(2x + 3) = (x - 3)(2x + 3). \]

The denominator of the first fraction in equation (17) easily factors using the difference of two squares pattern.

\[ x^2 - 16 = (x + 4)(x - 4) \]

The numerator of the second fraction in equation (17) is a quadratic trinomial, with \(ac = (2)(-12) = -24\). The integer pair \(-3\) and \(8\) have product \(-24\) and sum 5. Hence,

\[
\begin{align*}
2x^2 + 5x - 12 &= 2x^2 - 3x + 8x - 12 \\
&= x(2x - 3) + 4(2x - 3) \\
&= (x + 4)(2x - 3).
\end{align*}
\]

To factor the denominator of the last fraction in equation (17), first pull the greatest common factor (in this case \(x\)), then complete the factorization using the difference of two squares pattern.

\[
4x^3 - 9x = x(4x^2 - 9) = x(2x + 3)(2x - 3)
\]

We can now replace each numerator and denominator in equation (17) with its factorization, then cancel common factors.

\[
\begin{align*}
-\frac{2x^2 - 3x - 9}{x^2 - 16} \cdot \frac{2x^2 + 5x - 12}{4x^3 - 9x} &= -\frac{(x - 3)(2x + 3)}{(x + 4)(x - 4)} \cdot \frac{(x + 4)(2x - 3)}{x(2x + 3)(2x - 3)} \\
&= -\frac{x - 3}{x(x - 4)}
\end{align*}
\]

The last denominator has factors \(x\) and \(x - 4\), so \(x = 0\) and \(x = 4\) are restrictions. In the body of our work, the first fraction’s denominator has factors \(x + 4\) and \(x - 4\). We’ve seen the factor \(x - 4\) already, so only the factor \(x + 4\) adds a new restriction, \(x = -4\). Again, in the body of our work, the second fraction’s denominator has factors \(x\), \(2x + 3\), and \(2x - 3\), so we have added restrictions \(x = 0\), \(x = -3/2\), and \(x = 3/2\).

There’s one bit of trickery here that can easily be overlooked. In the body of our work, the second fraction’s numerator was originally a denominator before we inverted the fraction. So, we must consider what makes this numerator zero as well. Fortunately, the factors in this numerator are \(x + 4\) and \(2x - 3\) and we’ve already considered the restrictions produced by these factors.

Hence, for all values of \(x\), except the restricted values \(-4\), \(-3/2\), 0, \(3/2\), and 4, the left-hand side of

\[
\frac{9 + 3x - 2x^2}{x^2 - 16} \div \frac{4x^3 - 9x}{2x^2 + 5x - 12} = -\frac{x - 3}{x(x - 4)} \tag{18}
\]

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is identical to the right-hand side.

Again, this claim is easily checked by using a graphing calculator, as is partially evidenced (you’ll have to scroll downward to see the last restriction come into view) in the sequence of screen captures in Figure 4.

![Figure 4](image_url)

**Figure 4.** Using the table feature of the calculator to check the result in equation (18).

---

**Alternative Notation.** Note that the fractional expression $a/b$ means “$a$ divided by $b$,” so we can use this equivalent notation for $a \div b$. For example, the expression

$$\frac{9 + 3x - 2x^2}{x^2 - 16} \div \frac{4x^3 - 9x}{2x^2 + 5x - 12}$$

is equivalent to the expression

$$\frac{9 + 3x - 2x^2}{x^2 - 16} \cdot \frac{4x^3 - 9x}{2x^2 + 5x - 12}$$

Let’s look at an example of this notation in use.

**Example 21.** Given that

$$f(x) = \frac{x}{x + 3} \quad \text{and} \quad g(x) = \frac{x^2}{x + 3},$$

simplify both $f(x)g(x)$ and $f(x)/g(x)$.

First, the multiplication. There is no possible cancellation, so we simply multiply numerators and denominators.

$$f(x)g(x) = \frac{x}{x + 3} \cdot \frac{x^2}{x + 3} = \frac{x^3}{(x + 3)^2}.$$  

This result is valid for all values of $x$ except $-3$.

On the other hand,

$$\frac{f(x)}{g(x)} = \frac{\frac{x}{x + 3}}{\frac{x^2}{x + 3}} = \frac{x}{x + 3} \div \frac{x^2}{x + 3}.$$  

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When we “invert and multiply,” then cancel, we obtain

\[
\frac{f(x)}{g(x)} = \frac{x}{x + 3} \cdot \frac{x + 3}{x^2} = \frac{1}{x}.
\]

This result is valid for all values of \(x\) except \(-3\) and \(0\).
7.4 Exercises

In Exercises 1-10, reduce the product to a single fraction in lowest terms.

1. \( \frac{108}{14} \cdot \frac{6}{100} \)
2. \( \frac{75}{63} \cdot \frac{18}{45} \)
3. \( \frac{189}{56} \cdot \frac{12}{27} \)
4. \( \frac{45}{72} \cdot \frac{63}{64} \)
5. \( \frac{15}{36} \cdot \frac{28}{100} \)
6. \( \frac{189}{49} \cdot \frac{32}{25} \)
7. \( \frac{21}{100} \cdot \frac{125}{16} \)
8. \( \frac{21}{35} \cdot \frac{49}{45} \)
9. \( \frac{56}{20} \cdot \frac{98}{32} \)
10. \( \frac{27}{125} \cdot \frac{4}{12} \)

In Exercises 11-34, multiply and simplify. State all restrictions.

11. \( \frac{x + 6}{x^2 + 16x + 63} \cdot \frac{x^2 + 7x}{x + 4} \)
12. \( \frac{x^2 + 9x}{x^2 - 25} \cdot \frac{x^2 - x - 20}{-18 - 11x - x^2} \)
13. \( \frac{x^2 + 7x + 10}{x^2 - 1} \cdot \frac{-9 + 10x - x^2}{x^2 + 9x + 20} \)
14. \( \frac{x^2 + 5x}{x - 4} \cdot \frac{x - 2}{x^2 + 6x + 5} \)
15. \( \frac{x^2 - 5x}{x^2 + 2x - 48} \cdot \frac{x^2 + 11x + 24}{x^2 - x} \)
16. \( \frac{x^2 - 6x - 27}{x^2 + 10x + 24} \cdot \frac{x^2 + 13x + 42}{x^2 - 11x + 18} \)
17. \( \frac{-x - x^2}{x^2 - 9x + 8} \cdot \frac{x^2 - 4x + 3}{x^2 + 4x + 3} \)
18. \( \frac{x^2 - 12x + 35}{x^2 + 2x - 15} \cdot \frac{45 + 4x - x^2}{x^2 + x - 30} \)
19. \( \frac{x + 2}{7 - x} \cdot \frac{x^2 + x - 56}{x^2 + 7x + 6} \)
20. \( \frac{x^2 - 2x - 15}{x^2 + x} \cdot \frac{x^2 + 7x}{x^2 + 12x + 27} \)
21. \( \frac{x^2 - 9}{x^2 - 4x - 45} \cdot \frac{x - 6}{-3 - x} \)

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22. \[ \frac{x^2 - 12x + 27}{x - 4} \cdot \frac{x - 5}{x^2 - 18x + 81} \]

23. \[ \frac{x + 5}{x^2 + 12x + 32} \cdot \frac{x^2 - 2x - 24}{x + 7} \]

24. \[ \frac{x^2 - 36}{x^2 + 11x + 24} \cdot \frac{-8 - x}{x + 4} \]

25. \[ \frac{x - 5}{x^2 - 8x + 12} \cdot \frac{x^2 - 12x + 36}{x - 8} \]

26. \[ \frac{x^2 - 5x - 36}{x - 1} \cdot \frac{x - 5}{x^2 - 81} \]

27. \[ \frac{x^2 + 2x - 15}{x^2 - 10x + 16} \cdot \frac{x^2 - 7x + 10}{3x^2 + 13x - 10} \]

28. \[ \frac{5x^2 + 14x - 3}{x + 9} \cdot \frac{x - 7}{x^2 + 10x + 21} \]

29. \[ \frac{x^2 - 4}{x^2 + 2x - 63} \cdot \frac{x^2 + 6x - 27}{x^2 - 6x - 16} \]

30. \[ \frac{x^2 + 5x + 6}{x^2 - 3x} \cdot \frac{x^2 - 5x}{x^2 + 9x + 18} \]

31. \[ \frac{x - 1}{x^2 + 2x - 63} \cdot \frac{x^2 - 81}{x + 4} \]

32. \[ \frac{x^2 + 9x}{x^2 + 7x + 12} \cdot \frac{27 + 6x - x^2}{x^2 - 5x} \]

33. \[ \frac{5 - x}{x + 3} \cdot \frac{x^2 + 3x - 18}{2x^2 - 7x - 15} \]

34. \[ \frac{4x^2 + 21x + 5}{18 - 7x - x^2} \cdot \frac{x^2 + 11x + 18}{x^2 - 25} \]

In Exercises 35-58, divide and simplify. State all restrictions.

35. \[ \frac{x^2 - 14x + 48}{x^2 + 10x + 16} \div \frac{-24 + 11x - x^2}{x^2 - x - 72} \]

36. \[ \frac{x - 1}{x^2 - 14x + 48} \div \frac{x + 5}{x^2 - 3x - 18} \]

37. \[ \frac{x^2 - 1}{x^2 - 7x + 12} \div \frac{x^2 + 6x + 5}{-24 + 10x - x^2} \]

38. \[ \frac{x^2 - 13x + 42}{x^2 - 2x - 63} \div \frac{x^2 - x - 42}{x^2 + 8x + 7} \]

39. \[ \frac{x^2 - 25}{x + 1} \div \frac{5x^2 + 23x - 10}{x - 3} \]
40. \[
\frac{x^2 - 3x}{x^2 - 7x + 6} \div \frac{x^2 - 4x}{3x^2 - 11x - 42}
\]

41. \[
\frac{x^2 + 10x + 21}{x - 4} \div \frac{x^2 + 3x}{x + 8}
\]

42. \[
\frac{x^2 + 8x + 15}{x^2 - 14x + 45} \div \frac{x^2 + 11x + 30}{-30 + 11x - x^2}
\]

43. \[
\frac{x^2 - 6x - 16}{x^2 + x - 42} \div \frac{x^2 - 64}{x^2 + 12x + 35}
\]

44. \[
\frac{x^2 + 3x + 2}{x^2 - 9x + 18} \div \frac{x^2 + 7x + 6}{x^2 - 6x}
\]

45. \[
\frac{x^2 + 12x + 35}{x + 4} \div \frac{x^2 + 10x + 25}{x + 9}
\]

46. \[
\frac{x^2 - 8x + 7}{x^2 + 3x - 18} \div \frac{x^2 - 7x}{x^2 + 6x - 27}
\]

47. \[
\frac{x^2 + x - 30}{x^2 + 5x - 36} \div \frac{-6 - x}{x + 8}
\]

48. \[
\frac{2x - x^2}{x^2 - 15x + 54} \div \frac{x^2 + x}{x^2 - 11x + 30}
\]

49. \[
\frac{x^2 - 9x + 8}{x^2 - 9} \div \frac{x^2 - 8x}{-15 - 8x - x^2}
\]

50. \[
\frac{x + 5}{x^2 + 2x + 1} \div \frac{x - 2}{x^2 + 10x + 9}
\]

51. \[
\frac{x^2 - 4}{x + 8} \div \frac{x^2 - 10x + 16}{x + 3}
\]

52. \[
\frac{27 - 6x - x^2}{x^2 + 9x + 20} \div \frac{x^2 - 12x + 27}{x^2 + 5x}
\]

53. \[
\frac{x^2 + 5x + 6}{x^2 - 36} \div \frac{x - 7}{-6 - x}
\]

54. \[
\frac{2 - x}{x - 5} \div \frac{x^2 + 3x - 10}{x^2 - 14x + 48}
\]
55. \[
\frac{x + 3}{x^2 + 4x - 12} \div \frac{x - 4}{x^2 - 36}
\]

56. \[
\frac{x + 3}{x^2 - x - 2} \div \frac{x}{x^2 - 3x - 4}
\]

57. \[
\frac{x^2 - 11x + 28}{x^2 + 5x + 6} \div \frac{7x^2 - 30x + 8}{x^2 - x - 6}
\]

58. \[
\frac{x - 7}{3 - x} \div \frac{2x^2 + 3x - 5}{x^2 - 12x + 27}
\]

59. Let
\[
f(x) = \frac{x^2 - 7x + 10}{x^2 + 4x - 21}
\]
and
\[
g(x) = \frac{5x - x^2}{x^2 + 15x + 56}
\]
Compute \(f(x)/g(x)\) and simplify your answer.

60. Let
\[
f(x) = \frac{x^2 + 15x + 56}{x^2 - x - 20}
\]
and
\[
g(x) = \frac{-7 - x}{x + 1}
\]
Compute \(f(x)/g(x)\) and simplify your answer.

61. Let
\[
f(x) = \frac{x^2 + 12x + 35}{x^2 + 4x - 32}
\]
and
\[
g(x) = \frac{x^2 - 2x - 35}{x^2 + 8x}
\]
Compute \(f(x)/g(x)\) and simplify your answer.

62. Let
\[
f(x) = \frac{x^2 + 4x + 3}{x - 1}
\]
and
\[
g(x) = \frac{x^2 - 4x - 21}{x + 5}
\]
Compute \(f(x)/g(x)\) and simplify your answer.

63. Let
\[
f(x) = \frac{x^2 + x - 20}{x}
\]
and
\[
g(x) = \frac{x - 1}{x^2 - 2x - 35}
\]
Compute \(f(x)g(x)\) and simplify your answer.

64. Let
\[
f(x) = \frac{x^2 + 10x + 24}{x^2 - 13x + 42}
\]
and
\[
g(x) = \frac{x^2 - 6x - 7}{x^2 + 8x + 12}
\]
Compute \(f(x)g(x)\) and simplify your answer.
65. Let

\[ f(x) = \frac{x + 5}{-6 - x} \]

and

\[ g(x) = \frac{x^2 + 8x + 12}{x^2 - 49} \]

Compute \( f(x)g(x) \) and simplify your answer.

66. Let

\[ f(x) = \frac{8 - 7x - x^2}{x^2 - 8x - 9} \]

and

\[ g(x) = \frac{x^2 - 6x - 7}{x^2 - 6x + 5} \]

Compute \( f(x)g(x) \) and simplify your answer.
7.4 Answers

1. \(\frac{81}{175}\)  

23. Provided \(x \neq -8, -4, -7,\)  
\(\frac{(x + 5)(x - 6)}{(x + 8)(x + 7)}\)

2. \(\frac{3}{2}\)  

25. Provided \(x \neq 2, 6, 8,\)  
\(\frac{(x - 5)(x - 6)}{(x - 2)(x - 8)}\)

3. \(\frac{7}{60}\)  

27. Provided \(x \neq 2, 8, 2/3, -5,\)  
\(\frac{(x - 3)(x - 5)}{(3x - 2)(x - 8)}\)

4. \(\frac{105}{64}\)  

29. Provided \(x \neq -9, 7, 8, -2,\)  
\(\frac{(x - 2)(x - 3)}{(x - 7)(x - 8)}\)

5. \(\frac{343}{40}\)  

31. Provided \(x \neq 7, -9, -4,\)  
\(\frac{(x - 1)(x - 9)}{(x - 7)(x + 4)}\)

11. Provided \(x \neq -9, -7, -4,\)  
\(\frac{x(x + 6)}{(x + 9)(x + 4)}\)

13. Provided \(x \neq 1, -1, -4, -5,\)  
\(\frac{(x + 2)(x - 9)}{(x + 1)(x + 4)}\)

15. Provided \(x \neq -8, 6, 1, 0,\)  
\(\frac{(x - 5)(x + 3)}{(x - 6)(x - 1)}\)

17. Provided \(x \neq 1, 8, -3, -1,\)  
\(\frac{x(x - 3)}{(x - 8)(x + 3)}\)

19. Provided \(x \neq 7, -1, -6,\)  
\(\frac{(x + 2)(x + 8)}{(x + 1)(x + 6)}\)

21. Provided \(x \neq -3, -5, 9,\)  
\(\frac{(x - 3)(x - 6)}{(x + 5)(x - 9)}\)

23. Provided \(x \neq -8, -4, -7,\)  
\(\frac{(x + 5)(x - 6)}{(x + 8)(x + 7)}\)

25. Provided \(x \neq 2, 6, 8,\)  
\(\frac{(x - 5)(x - 6)}{(x - 2)(x - 8)}\)

27. Provided \(x \neq 2, 8, 2/3, -5,\)  
\(\frac{(x - 3)(x - 5)}{(3x - 2)(x - 8)}\)

29. Provided \(x \neq -9, 7, 8, -2,\)  
\(\frac{(x - 2)(x - 3)}{(x - 7)(x - 8)}\)

31. Provided \(x \neq 7, -9, -4,\)  
\(\frac{(x - 1)(x - 9)}{(x - 7)(x + 4)}\)

33. Provided \(x \neq -3, -3/2, 5,\)  
\(\frac{(x + 6)(x - 3)}{(2x + 3)(x + 3)}\)

35. Provided \(x \neq -8, -2, 9, 3, 8,\)  
\(\frac{(x - 6)(x - 9)}{(x + 2)(x - 3)}\)

37. Provided \(x \neq 4, 3, 6, -5, -1,\)  
\(\frac{(x - 1)(x - 6)}{(x - 3)(x + 5)}\)

39. Provided \(x \neq -1, 2/5, -5, 3,\)  
\(\frac{(x - 5)(x - 3)}{(5x - 2)(x + 1)}\)
41. Provided $x \neq 4, 0, -3, -8,$
\[ \frac{(x + 7)(x + 8)}{x(x - 4)} \]

43. Provided $x \neq -7, 6, -5, -8, 8,$
\[ \frac{(x + 2)(x + 5)}{(x - 6)(x + 8)} \]

45. Provided $x \neq -4, -5, -9,$
\[ \frac{(x + 7)(x + 9)}{(x + 4)(x + 5)} \]

47. Provided $x \neq 4, -9, -8, -6,$
\[ \frac{-(x - 5)(x + 8)}{(x - 4)(x + 9)} \]

49. Provided $x \neq -3, 3, -5, 0, 8,$
\[ \frac{-(x - 1)(x + 5)}{x(x - 3)} \]

51. Provided $x \neq -8, 8, 2, -3,$
\[ \frac{(x + 2)(x + 3)}{(x + 8)(x - 8)} \]

53. Provided $x \neq 6, -6, 7,$
\[ \frac{-(x + 2)(x + 3)}{(x - 6)(x - 7)} \]

55. Provided $x \neq 2, -6, 4, 6,$
\[ \frac{(x + 3)(x - 6)}{(x - 2)(x - 4)} \]

57. Provided $x \neq -2, -3, 3, 2/7, 4,$
\[ \frac{(x - 7)(x - 3)}{(7x - 2)(x + 3)} \]
7.5 Sums and Differences of Rational Functions

In this section we concentrate on finding sums and differences of rational expressions. However, before we begin, we need to review some fundamental ideas and technique.

First and foremost is the concept of the multiple of an integer. This is best explained with a simple example. The multiples of 8 is the set of integers \( \{8k : k \text{ is an integer}\} \). In other words, if you multiply 8 by \( 0, \pm 1, \pm 2, \pm 3, \pm 4 \), etc., you produce what is known as the multiples of 8.

Multiples of 8 are: \( 0, \pm 8, \pm 16, \pm 24, \pm 32 \), etc.

However, for our purposes, only the positive multiples are of interest. So we will say:

Multiples of 8 are: \( 8, 16, 24, 32, 40, 48, 56, 64, 72 \ldots \)

Similarly, we can list the positive multiples of 6.

Multiples of 6 are: \( 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72 \ldots \)

We’ve framed those numbers that are multiples of both 8 and 6. These are called the common multiples of 8 and 6.

Common multiples of 8 and 6 are: \( 24, 48, 72 \ldots \)

The smallest of this list of common multiples of 8 and 6 is called the least common multiple of 8 and 6. We will use the following notation to represent the least common multiple of 8 and 6: \( \text{LCM}(8, 6) \).

Hopefully, you will now feel comfortable with the following definition.

**Definition 1.** Let \( a \) and \( b \) be integers. The least common multiple of \( a \) and \( b \), denoted \( \text{LCM}(a, b) \), is the smallest positive multiple that \( a \) and \( b \) have in common.

For larger numbers, listing multiples until you find one in common can be impractical and time consuming. Let’s find the least common multiple of 8 and 6 a second time, only this time let’s use a different technique.

First, write each number as a product of primes in exponential form.

\[
\begin{align*}
8 &= 2^3 \\
6 &= 2 \cdot 3
\end{align*}
\]

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Here’s the rule.

**A Procedure to Find the LCM.** To find the LCM of two integers, proceed as follows.

1. Express the prime factorization of each integer in exponential format.
2. To find the least common multiple, write down every prime number that appears, then affix the largest exponent of that prime that appears.

In our example, the primes that occur are 2 and 3. The highest power of 2 that occurs is $2^3$. The highest power of 3 that occurs is $3^1$. Thus, the LCM(8, 6) is

$$\text{LCM}(8, 6) = 2^3 \cdot 3^1 = 24.$$  

Note that this result is identical to the result found above by listing all common multiples and choosing the smallest.

Let’s try a harder example.

**Example 2.** *Find the least common multiple of 24 and 36.*

Using the first technique, we list the multiples of each number, framing the multiples in common.

- Multiples of 24: 24, 48, 72, 96, 120, 144, 168, ...
- Multiples of 36: 36, 72, 108, 144, 180, ...

The multiples in common are 72, 144, etc., and the least common multiple is $\text{LCM}(24, 36) = 72$.

Now, let’s use our second technique to find the least common multiple (LCM). First, express each number as a product of primes in exponential format.

$$24 = 2^3 \cdot 3$$
$$36 = 2^2 \cdot 3^2$$

To find the least common multiple, write down every prime that occurs and affix the highest power of that prime that occurs. Thus, the highest power of 2 that occurs is $2^3$, and the highest power of 3 that occurs is $3^2$. Thus, the least common multiple is

$$\text{LCM}(24, 36) = 2^3 \cdot 3^2 = 8 \cdot 9 = 72.$$
Addition and Subtraction Defined

Imagine a pizza that has been cut into 12 equal slices. Then, each slice of pizza represents $\frac{1}{12}$ of the entire pizza.

If Jimmy eats 3 slices, then he has consumed $\frac{3}{12}$ of the entire pizza. If Margaret eats 2 slices, then she has consumed $\frac{2}{12}$ of the entire pizza. It’s clear that together they have consumed

$$\frac{3}{12} + \frac{2}{12} = \frac{5}{12}$$

of the pizza. It would seem that adding two fractions with a common denominator is as simple as eating pizza! Hopefully, the following definition will seem reasonable.

**Definition 3.** To add two fractions with a common denominator, such as $\frac{a}{c}$ and $\frac{b}{c}$, add the numerators and divide by the common denominator. In symbols,

$$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}.$$  

Note how this definition agrees precisely with our pizza consumption discussed above. Here are some examples of adding fractions having common denominators.

$$\frac{5}{21} + \frac{3}{21} = \frac{5 + 3}{21} = \frac{8}{21}$$

$$\frac{2}{x + 2} + \frac{x - 3}{x + 2} = \frac{2 + (x - 3)}{x + 2} = \frac{2 + x - 3}{x + 2} = \frac{x - 1}{x + 2}$$

Subtraction works in much the same way as does addition.

**Definition 4.** To subtract two fractions with a common denominator, such as $\frac{a}{c}$ and $\frac{b}{c}$, subtract the numerators and divide by the common denominator. In symbols,

$$\frac{a}{c} - \frac{b}{c} = \frac{a - b}{c}.$$  

Here are some examples of subtracting fractions already having common denominators.

$$\frac{5}{21} - \frac{3}{21} = \frac{5 - 3}{21} = \frac{2}{21}$$

$$\frac{2}{x + 2} - \frac{x - 3}{x + 2} = \frac{2 - (x - 3)}{x + 2} = \frac{2 - x + 3}{x + 2} = \frac{5 - x}{x + 2}$$
In the example on the right, note that it is extremely important to use grouping symbols when subtracting numerators. Note that the minus sign in front of the parenthetical expression changes the sign of each term inside the parentheses.

There are times when a sign change will provide a common denominator.

Example 5. Simplify

\[
\frac{x}{x-3} - \frac{2}{3-x}. \quad (6)
\]

State all restrictions.

At first glance, it appears that we do not have a common denominator. On second glance, if we make a sign change on the second fraction, it might help. So, on the second fraction, let’s negate the denominator and fraction bar to obtain

\[
\frac{x}{x-3} - \frac{2}{3-x} = \frac{x}{x-3} + \frac{2}{x-3} = \frac{x+2}{x-3}. \quad (7)
\]

The denominators \(x - 3\) or \(3 - x\) are zero when \(x = 3\). Hence, 3 is a restricted value. For all other values of \(x\), the left-hand side of

\[
\frac{x}{x-3} - \frac{2}{3-x} = \frac{x+2}{x-3}.
\]

is identical to the right-hand side.

This is easily tested using the table utility on the graphing calculator, as shown in the sequence of screenshots in Figure 1. First load the left- and right-hand sides of equation (7) into Y1 and Y2 in the Y= menu of your graphing calculator, as shown in Figure 1(a). Press 2nd TBLSET and make the changes shown in Figure 1(b). Press 2nd TABLE to produce the table shown in Figure 1(c). Note the ERR (error) message at the restriction \(x = 3\), but note also the agreement of Y1 and Y2 for all other values of \(x\).

Figure 1. Using the table feature of the graphing calculator to check the result in equation (7).
Equivalent Fractions

If you slice a pizza into four equal pieces, then consume two of the four slices, you’ve consumed half of the pizza. This motivates the fact that
\[ \frac{1}{2} = \frac{2}{4}. \]

Indeed, if you slice the pizza into six equal pieces, then consume three slices, you’ve consumed half of the pizza, so it’s fair to say that \( \frac{3}{6} = \frac{1}{2} \). Indeed, all of the following fractions are equivalent:

\[
\begin{align*}
\frac{1}{2} &= \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{6}{12} = \frac{7}{14} = \cdots
\end{align*}
\]

A more formal way to demonstrate that \( \frac{1}{2} \) and \( \frac{7}{14} \) are equal is to start with the fact that \( \frac{1}{2} = \frac{1}{2} \times 1 \), then replace \( 1 \) with \( \frac{7}{7} \) and multiply.

\[
\frac{1}{2} = \frac{1}{2} \times 1 = \frac{1}{2} \times \frac{7}{7} = \frac{7}{14}
\]

Here’s another example of this principle in action, only this time we replace \( 1 \) with \( \frac{(x-2)}{(x-2)} \).

\[
\frac{3}{x+2} = \frac{3}{x+2} \cdot 1 = \frac{3}{x+2} \cdot \frac{x-2}{x-2} = \frac{3(x-2)}{(x+2)(x-2)}
\]

In the next example we replace \( 1 \) with \( \frac{(x(x-3))}{(x-3)} \).

\[
\frac{2}{x-4} = \frac{2}{x-4} \cdot 1 = \frac{2}{x-4} \cdot \frac{x(x-3)}{x-3} = \frac{2x(x-3)}{(x-4)(x-3)}
\]

Now, let’s apply the concept of equivalent fractions to add and subtract fractions with different denominators.

Adding and Subtracting Fractions with Different Denominators

In this section we show our readers how to add and subtract fractions having different denominators. For example, suppose we are asked to add the following fractions.

\[ \frac{5}{12} + \frac{5}{18} \quad (8) \]

First, we must find a “common denominator.” Fortunately, the machinery to find the “common denominator” is already in place. It turns out that the least common denominator for 12 and 18 is the least common multiple of 12 and 18.

\[
\begin{align*}
18 &= 2 \cdot 3^2 \\
12 &= 2^2 \cdot 3 \\
\text{LCD}(12, 18) &= 2^2 \cdot 3^2 = 36
\end{align*}
\]
The next step is to create equivalent fractions using the LCD as the denominator. So, in the case of 5/12,

\[
\frac{5}{12} = \frac{5}{12} \cdot \frac{1}{1} = \frac{5}{12} \cdot \frac{3}{3} = \frac{15}{36}.
\]

In the case of 5/18,

\[
\frac{5}{18} = \frac{5}{18} \cdot \frac{1}{1} = \frac{5}{18} \cdot \frac{2}{2} = \frac{10}{36}.
\]

If we replace the fractions in equation (8) with their equivalent fractions, we can then add the numerators and divide by the common denominator, as in

\[
\frac{5}{12} + \frac{5}{18} = \frac{15}{36} + \frac{10}{36} = \frac{25}{36}.
\]

Let’s examine a method of organizing the work that is more compact. Consider the following arrangement, where we’ve used color to highlight the form of 1 required to convert the fractions to equivalent fractions with a common denominator of 36.

\[
\frac{5}{12} + \frac{5}{18} = \frac{5}{12} \cdot \frac{3}{3} + \frac{5}{18} \cdot \frac{2}{2} = \frac{15}{36} + \frac{10}{36} = \frac{25}{36}.
\]

Let’s look at a more complicated example.

**Example 9.** Simplify the expression

\[
\frac{x + 3}{x + 2} - \frac{x + 2}{x + 3}.
\]

State all restrictions.

The denominators are already factored. If we take each factor that appears to the highest exponential power that appears, our least common denominator is \((x+2)(x+3)\). Our first task is to make equivalent fractions having this common denominator.

\[
\frac{x + 3}{x + 2} - \frac{x + 2}{x + 3} = \frac{x + 3}{x + 2} \cdot \frac{x + 3}{x + 3} - \frac{x + 2}{x + 3} \cdot \frac{x + 2}{x + 2} = \frac{x^2 + 6x + 9}{(x + 2)(x + 3)} - \frac{x^2 + 4x + 4}{(x + 2)(x + 3)}
\]

Now, subtract the numerators and divide by the common denominator.
\[
\frac{x + 3}{x + 2} - \frac{x + 2}{x + 3} = \frac{(x^2 + 6x + 9) - (x^2 + 4x + 4)}{(x + 2)(x + 3)} \\
= \frac{x^2 + 6x + 9 - x^2 - 4x - 4}{(x + 2)(x + 3)} \\
= \frac{2x + 5}{(x + 2)(x + 3)}
\]

Note the use of parentheses when we subtracted the numerators. Note further how the minus sign negates each term in the parenthetical expression that follows the minus sign.

**Tip 11.** Always use grouping symbols when subtracting the numerators of fractions.

In the final answer, the factors \(x + 2\) and \(x + 3\) in the denominator are zero when \(x = -2\) or \(x = -3\). These are the restrictions. No other denominators, in the original problem or in the body of our work, provide additional restrictions.

Thus, for all values of \(x\), except the restricted values \(-2\) and \(-3\), the left-hand side of

\[
\frac{x + 3}{x + 2} - \frac{x + 2}{x + 3} = \frac{2x + 5}{(x + 2)(x + 3)} \tag{12}
\]

is identical to the right-hand side. This claim is easily tested on the graphing calculator which is evidenced in the sequence of screen captures in Figure 2. Note the ERR (error) message at each restricted value of \(x\) in Figure 2(c), but also note the agreement of \(Y1\) and \(Y2\) for all other values of \(x\).

![Figure 2](image-url)

**Figure 2.** Using the table feature of the graphing calculator to check the result in equation (12).
Let’s look at another example.

**Example 13.** Simplify the expression

\[
\frac{4}{x^2 + 6x + 5} - \frac{2}{x^2 + 8x + 15}.
\]

State all restrictions.

First, factor each denominator.

\[
\frac{4}{x^2 + 6x + 5} - \frac{2}{x^2 + 8x + 15} = \frac{4}{(x + 1)(x + 5)} - \frac{2}{(x + 3)(x + 5)}.
\]

The least common denominator, or least common multiple (LCM), requires that we write down each factor that occurs, then affix the highest power of that factor that occurs. Because all factors in the denominators are raised to an understood power of one, the LCD (least common denominator) or LCM is \((x + 1)(x + 5)(x + 3)\).

Next, we make equivalent fractions having this common denominator.

\[
\frac{4}{x^2 + 6x + 5} - \frac{2}{x^2 + 8x + 15} = \frac{4}{(x + 1)(x + 5)} \cdot \frac{x + 3}{x + 3} - \frac{2}{(x + 3)(x + 5)} \cdot \frac{x + 1}{x + 1}.
\]

Subtract the numerators and divide by the common denominator. Be sure to use grouping symbols, particularly with the minus sign that is in play.

\[
\frac{4}{x^2 + 6x + 5} - \frac{2}{x^2 + 8x + 15} = \frac{(4x + 12) - (2x + 2)}{(x + 3)(x + 5)(x + 1)} = \frac{4x + 12 - 2x - 2}{(x + 3)(x + 5)(x + 1)} = \frac{2x + 10}{(x + 3)(x + 5)(x + 1)}
\]

Finally, we should always make sure that our answer is reduced to lowest terms. With that thought in mind, we factor the numerator in hopes that we can get a common factor to cancel.

\[
\frac{4}{x^2 + 6x + 5} - \frac{2}{x^2 + 8x + 15} = \frac{2(x + 5)}{(x + 3)(x + 5)(x + 1)} = \frac{2}{(x + 3)(x + 1)}.
\]

The denominators have factors of \(x + 3\), \(x + 5\) and \(x + 1\), so the restrictions are \(x = -3\), \(x = -5\), and \(x = -1\), respectively. For all other values of \(x\), the left-hand side of

\[
\frac{4}{x^2 + 6x + 5} - \frac{2}{x^2 + 8x + 15} = \frac{2}{(x + 3)(x + 1)} \quad (14)
\]
is identical to its right-hand side. Again, this is easily tested using the table feature of the graphing calculator, as shown in the screenshots in Figure 3. Again, note the ERR (error) messages at each restricted value of x, but also note that Y1 and Y2 agree for all other values of x.

![Figure 3](image-url)

**Figure 3.** Using the table feature of the graphing calculator to check the result in equation (14).

---

Let’s look at another example.

**Example 15.** Simplify the expression

$$\frac{x - 3}{x^2 - 1} + \frac{1}{x + 1} - \frac{1}{1 - x}.$$  

State all restrictions.

First, factor all denominators.

$$\frac{x - 3}{x^2 - 1} + \frac{1}{x + 1} - \frac{1}{1 - x} = \frac{x - 3}{(x + 1)(x - 1)} + \frac{1}{x + 1} - \frac{1}{1 - x}.$$  

If we’re not careful, we might be tempted to take one of each factor and use \((x + 1)(x - 1)(1 - x)\) as a common denominator. However, let’s first make two negations of the last of the three fractions on the right, negating the fraction bar and denominator to get

$$\frac{x - 3}{x^2 - 1} + \frac{1}{x + 1} - \frac{1}{1 - x} = \frac{x - 3}{(x + 1)(x - 1)} + \frac{1}{x + 1} + \frac{1}{x - 1}.$$  

Now we can see that a common denominator of \((x + 1)(x - 1)\) will suffice. Let’s make equivalent fractions with this common denominator.

$$\frac{x - 3}{x^2 - 1} + \frac{1}{x + 1} - \frac{1}{1 - x} = \frac{x - 3}{(x + 1)(x - 1)} + \frac{1}{x + 1} + \frac{1}{x + 1}.$$  

Add the numerators and divide by the common denominator. Even though grouping symbols are not as critical in this problem (because of the plus signs), we still think it good practice to use them.
\[
\frac{x - 3}{x^2 - 1} + \frac{1}{x + 1} - \frac{1}{1 - x} = \frac{(x - 3) + (x - 1) + (x + 1)}{(x + 1)(x - 1)} \\
= \frac{3x - 3}{(x + 1)(x - 1)}
\]

Finally, always make sure that your final answer is reduced to lowest terms. With that thought in mind, we factor the numerator in hopes that we can get a common factor to cancel.

\[
\frac{x - 3}{x^2 - 1} + \frac{1}{x + 1} - \frac{1}{1 - x} = \frac{3(x - 1)}{(x + 1)(x - 1)} \\
= \frac{3(x - 1)}{(x + 1)(x - 1)} \\
= \frac{3}{x + 1}
\]

The factors \(x + 1\) and \(x - 1\) in the denominator produce restrictions \(x = -1\) and \(x = 1\), respectively. However, for all other values of \(x\), the left-hand side of

\[
\frac{x - 3}{x^2 - 1} + \frac{1}{x + 1} - \frac{1}{1 - x} = \frac{3}{x + 1} \tag{16}
\]

is identical to the right-hand side. Again, this is easily checked on the graphing calculator as shown in the sequence of screenshots in Figure 4.

![Figure 4](image.png)

Figure 4. Using the table feature of the graphing calculator to check the result in equation (16).

Again, note the ERR (error) messages at each restriction, but also note that the values of \(Y1\) and \(Y2\) agree for all other values of \(x\).
Let’s look at an example using function notation.

Example 17. If the function \( f \) and \( g \) are defined by the rules

\[
    f(x) = \frac{x}{x + 2} \quad \text{and} \quad g(x) = \frac{1}{x},
\]

simplify \( f(x) - g(x) \).

First,

\[
    f(x) - g(x) = \frac{x}{x + 2} - \frac{1}{x}.
\]

Note how tempting it would be to cancel. However, canceling would be an error in this situation, because subtraction requires a common denominator.

\[
    f(x) - g(x) = \frac{x}{x + 2} \cdot \frac{x}{x} - \frac{1}{x} \cdot \frac{x + 2}{x + 2}
\]

\[
    = \frac{x^2}{x(x + 2)} - \frac{x + 2}{x(x + 2)}
\]

Subtract numerators and divide by the common denominator. This requires that we “distribute” the minus sign.

\[
    f(x) - g(x) = \frac{x^2 - (x + 2)}{x(x + 2)}
\]

\[
    = \frac{x^2 - x - 2}{x(x + 2)}
\]

This result is valid for all values of \( x \) except 0 and \(-2\). We leave it to our readers to verify that this result is reduced to lowest terms. You might want to check the result on your calculator as well.
7.5 **Exercises**

In **Exercises 1-16**, add or subtract the rational expressions, as indicated, and simplify your answer. State all restrictions.

1. \[ \frac{7x^2 - 49x}{x - 6} + \frac{42}{x - 6} \]
2. \[ \frac{2x^2 - 110}{x - 7} - \frac{12}{7 - x} \]
3. \[ \frac{27x - 9x^2}{x + 3} + \frac{162}{x + 3} \]
4. \[ \frac{2x^2 - 28}{x + 2} - \frac{10x}{x + 2} \]
5. \[ \frac{4x^2 - 8}{x - 4} + \frac{56}{4 - x} \]
6. \[ \frac{4x^2}{x - 2} - \frac{36x - 56}{x - 2} \]
7. \[ \frac{9x^2}{x - 1} + \frac{72x - 63}{1 - x} \]
8. \[ \frac{5x^2 + 30}{x - 6} - \frac{35x}{x - 6} \]
9. \[ \frac{4x^2 - 60x}{x - 7} + \frac{224}{x - 7} \]
10. \[ \frac{3x^2}{x - 7} - \frac{63 - 30x}{7 - x} \]
11. \[ \frac{3x^2}{x - 2} - \frac{48 - 30x}{2 - x} \]
12. \[ \frac{4x^2 - 164}{x - 6} - \frac{20}{6 - x} \]
13. \[ \frac{9x^2}{x - 2} - \frac{81x - 126}{x - 2} \]
14. \[ \frac{9x^2}{x - 8} + \frac{144x - 576}{8 - x} \]
15. \[ \frac{3x^2 - 12}{x - 3} + \frac{15}{3 - x} \]
16. \[ \frac{7x^2}{x - 9} - \frac{112x - 441}{x - 9} \]

In **Exercises 17-34**, add or subtract the rational expressions, as indicated, and simplify your answer. State all restrictions.

17. \[ \frac{3x}{x^2 - 6x + 5} \]
18. \[ \frac{7x}{x^2 - 4x} + \frac{28}{x^2 - 12x + 32} \]
19. \[ \frac{9x}{x^2 + 4x - 12} - \frac{54}{x^2 + 20x + 84} \]
20. \[ \frac{9x}{x^2 - 25} - \frac{45}{x^2 + 20x + 75} \]
21. \[ \frac{5x}{x^2 - 21x + 98} - \frac{35}{7x - x^2} \]
22. \[ \frac{7x}{7x - x^2} + \frac{147}{x^2 + 7x - 98} \]
23. \[ \frac{-7x}{x^2 - 8x + 15} - \frac{35}{x^2 - 12x + 35} \]
24. \[ \frac{-6x}{x^2 + 2x} + \frac{12}{x^2 + 6x + 8} \]
25. \[ \frac{-9x}{x^2 - 12x + 32} - \frac{36}{x^2 - 4x} \]
26. \[ \frac{5x}{x^2 - 12x + 32} - \frac{20}{4x - x^2} \]
27. \[
\frac{6x}{x^2 - 21x + 98} - \frac{42}{7x - x^2}
\]

28. \[
\frac{-2x}{x^2 - 3x - 10} + \frac{4}{x^2 + 11x + 18}
\]

29. \[
\frac{-9x}{x^2 - 6x + 8} - \frac{18}{x^2 - 2x}
\]

30. \[
\frac{6x}{5x - x^2} + \frac{90}{x^2 + 5x - 50}
\]

31. \[
\frac{8x}{5x - x^2} + \frac{120}{x^2 + 5x - 50}
\]

32. \[
\frac{-5x}{x^2 + 5x} + \frac{25}{x^2 + 15x + 50}
\]

33. \[
\frac{-5x}{x^2 + x - 30} + \frac{30}{x^2 + 23x + 102}
\]

34. \[
\frac{9x}{x^2 + 12x + 32} - \frac{36}{x^2 + 4x}
\]

35. Let \[f(x) = \frac{8x}{x^2 + 6x + 8}\]
and \[g(x) = \frac{16}{x^2 + 2x}\]
Compute \(f(x) - g(x)\) and simplify your answer.

36. Let \[f(x) = \frac{-7x}{x^2 + 8x + 12}\]
and \[g(x) = \frac{42}{x^2 + 16x + 60}\]
Compute \(f(x) + g(x)\) and simplify your answer.

37. Let \[f(x) = \frac{11x}{x^2 + 12x + 32}\]
and \[g(x) = \frac{44}{-4x - x^2}\]
Compute \(f(x) + g(x)\) and simplify your answer.

38. Let \[f(x) = \frac{8x}{x^2 - 6x}\]
and \[g(x) = \frac{48}{x^2 - 18x + 72}\]
Compute \(f(x) + g(x)\) and simplify your answer.

39. Let \[f(x) = \frac{4x}{-x - x^2}\]
and \[g(x) = \frac{4}{x^2 + 3x + 2}\]
Compute \(f(x) + g(x)\) and simplify your answer.

40. Let \[f(x) = \frac{5x}{x^2 - x - 12}\]
and \[g(x) = \frac{15}{x^2 + 13x + 30}\]
Compute \(f(x) - g(x)\) and simplify your answer.
7.5 Answers

1. \(7(x - 1), \text{ provided } x \neq 6\).
2. \(-9(x - 6), \text{ provided } x \neq -3\).
3. \(4(x + 4), \text{ provided } x \neq 4\).
4. \(9(x - 7), \text{ provided } x \neq 1\).
5. \(4(x - 8), \text{ provided } x \neq 7\).
6. \(3(x - 8), \text{ provided } x \neq 2\).
7. \(9(x - 7), \text{ provided } x \neq 2\).
8. \(3(x + 3), \text{ provided } x \neq 3\).
9. \(Provided \ x \neq 5, 1, 9, \frac{3(x + 1)}{(x - 1)(x - 9)}\)
10. \(Provided \ x \neq -6, 2, -14, \frac{9(x + 2)}{(x - 2)(x + 14)}\)
11. \(Provided \ x \neq \frac{7, 14, 0,}{6(x + 14)}\)
12. \(Provided \ x \neq 2, 4, 0, \frac{-9(x + 4)}{x(x - 4)}\)
13. \(Provided \ x \neq 5, 0, -10, \frac{-8}{x + 10}\)
14. \(Provided \ x \neq -6, 5, -17, \frac{-5(x + 5)}{(x - 5)(x + 17)}\)
15. \(Provided \ x \neq -2, -4, 0, \frac{8(x - 4)}{x(x + 4)}\)
16. \(Provided \ x \neq -4, -8, 0, \frac{11(x - 8)}{x(x + 8)}\)
17. \(Provided \ x \neq 7, 14, 0, \frac{5(x + 14)}{x(x - 14)}\)
18. \(Provided \ x \neq -4, -8, 0, \frac{-9(x + 8)}{x(x - 8)}\)
In this section we learn how to simplify what are called complex fractions, an example of which follows.

\[
\frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{4} + \frac{2}{3}}
\]  \(1\)

Note that both the numerator and denominator are fraction problems in their own right, lending credence to why we refer to such a structure as a “complex fraction.”

There are two very different techniques we can use to simplify the complex fraction (1). The first technique is a “natural” choice.

**Simplifying Complex Fractions — First Technique.** To simplify a complex fraction, proceed as follows:

1. Simplify the numerator.
2. Simplify the denominator.
3. Simplify the division problem that remains.

Let’s follow this outline to simplify the complex fraction (1). First, add the fractions in the numerator as follows.

\[
\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}
\]  \(2\)

Secondly, add the fractions in the denominator as follows.

\[
\frac{1}{4} + \frac{2}{3} = \frac{3}{12} + \frac{8}{12} = \frac{11}{12}
\]  \(3\)

Substitute the results from (2) and (3) into the numerator and denominator of (1), respectively.

\[
\frac{\frac{5}{6}}{\frac{11}{12}}
\]  \(4\)

The right-hand side of (4) is equivalent to

\[
\frac{5}{6} \div \frac{11}{12}
\]

This is a division problem, so invert and multiply, factor, then cancel common factors.

---

16 Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/
Here is an arrangement of the work, from start to finish, presented without comment. This is a good template to emulate when doing your homework.

\[
\begin{align*}
\frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{4} + \frac{2}{3}} &= \frac{5 \cdot \frac{12}{6}}{11} \\
&= \frac{5 \cdot 2 \cdot 3}{11} \\
&= \frac{10}{11}
\end{align*}
\]

Now, let’s look at a second approach to the problem. We saw that simplifying the numerator in (2) required a common denominator of 6. Simplifying the denominator in (3) required a common denominator of 12. So, let’s choose another common denominator, this one a common denominator for both numerator and denominator, namely, 12. Now, multiply top and bottom (numerator and denominator) of the complex fraction (1) by 12, as follows.

\[
\begin{align*}
\frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{4} + \frac{2}{3}} &= \frac{\left(\frac{1}{2} + \frac{1}{3}\right) 12}{11} \\
&= \frac{5 \cdot 2 \cdot 3}{11} \\
&= \frac{10}{11}
\end{align*}
\]

Distribute the 12 in both numerator and denominator and simplify.
Let’s summarize this second technique.

**Simplifying Complex Fractions — Second Technique.** To simplify a complex fraction, proceed as follows:

1. Find a common denominator for both numerator and denominator.
2. Clear fractions from the numerator and denominator by multiplying each by the common denominator found in the first step.

Note that for this particular problem, the second method is much more efficient. It saves both space and time and is more aesthetically pleasing. It is the technique that we will favor in the rest of this section.

Let’s look at another example.

**Example 6.** Use both the First and Second Techniques to simplify the expression

\[
\frac{1}{x} - 1 \quad \frac{\frac{1}{x} - 1}{1 - \frac{1}{x^2}}.
\]

State all restrictions.

Let’s use the first technique, simplifying numerator and denominator separately before dividing. First, make equivalent fractions with a common denominator for the subtraction problem in the numerator of (7) and simplify. Do the same for the denominator.

\[
\frac{1}{x} - 1 = \frac{1 - x}{x - x^2}.
\]

Next, invert and multiply, then factor.

\[
\frac{1}{x} - 1 = \frac{1 - x}{x} \cdot \frac{x^2 - 1}{x^2} = \frac{1 - x}{x} \cdot \frac{x^2}{(x + 1)(x - 1)}.
\]

Let’s invoke the sign change rule and negate two parts of the fraction \((1 - x)/x\), numerator and fraction bar, then cancel the common factors.
\[
\frac{1}{x} - \frac{1}{1 - \frac{1}{x^2}} = \frac{x - 1}{x} \cdot \frac{x^2}{(x + 1)(x - 1)} = \frac{x}{x^2} \cdot \frac{x}{x} \cdot \frac{x}{x^2} = \frac{x}{x} \cdot \frac{x}{x} 
\]

Hence,
\[
\frac{1}{x} - \frac{1}{1 - \frac{1}{x^2}} = -\frac{x}{x + 1}.
\]

Now, let’s try the problem a second time, multiplying numerator and denominator by \(x^2\) to clear fractions from both the numerator and denominator.
\[
\frac{1}{x} - \frac{1}{1 - \frac{1}{x^2}} = \frac{\left(\frac{1}{x} - 1\right) x^2}{\left(1 - \frac{1}{x^2}\right) x^2} = \frac{\left(\frac{1}{x}\right) x^2 - (1) x^2}{(1)x^2 - \left(\frac{1}{x^2}\right) x^2} = \frac{x - x^2}{x^2 - 1}
\]

The order in the numerator of the last fraction intimates that a sign change would be helpful. Negate the numerator and fraction bar, factor, then cancel common factors.
\[
\frac{1}{x} - \frac{1}{1 - \frac{1}{x^2}} = -\frac{x^2 - x}{x^2 - 1} = -\frac{x(x - 1)}{(x + 1)(x - 1)} = -\frac{x - 1}{(x + 1)(x - 1)} = -\frac{x}{x + 1}
\]

This is precisely the same answer found with the first technique. To list the restrictions, we must make sure that no values of \(x\) make any denominator equal to zero, at the beginning of the problem, in the body of our work, or in the final answer.

In the original problem, if \(x = 0\), then both \(1/x\) and \(1/x^2\) are undefined, so \(x = 0\) is a restriction. In the body of our work, the factors \(x + 1\) and \(x - 1\) found in various denominators make \(x = -1\) and \(x = 1\) restrictions. No other denominators supply restrictions that have not already been listed. Hence, for all \(x\) other than \(-1\), 0, and 1, the left-hand side of
\[
\frac{1}{x} - \frac{1}{1 - \frac{1}{x^2}} = -\frac{x}{x + 1}
\]

is identical to the right-hand side. Again, the calculator’s table utility provides ample evidence of this fact in the screenshots shown in Figure 1.

Note the \text{ERR} (error) messages at each of the restricted values of \(x\), but also note the perfect agreement of \(Y1\) and \(Y2\) at all other values of \(x\).

Let’s look at another example, an important example involving function notation.

Version: Fall 2007
Example 9. Given that

\[ f(x) = \frac{1}{x}, \]

simplify the expression

\[ \frac{f(x) - f(2)}{x - 2}. \]

List all restrictions.

Remember, \( f(2) \) means substitute 2 for \( x \). Because \( f(x) = 1/x \), we know that \( f(2) = 1/2 \), so

\[ \frac{f(x) - f(2)}{x - 2} = \frac{1 - \frac{1}{2}}{x - 2}. \]

To clear the fractions from the numerator, we’d use a common denominator of \( 2x \). There are no fractions in the denominator that need clearing, so the common denominator for numerator and denominator is \( 2x \). Multiply numerator and denominator by \( 2x \).

\[ \frac{f(x) - f(2)}{x - 2} = \frac{\left( \frac{1}{x} - \frac{1}{2} \right) 2x}{(x - 2)2x} = \frac{\left( \frac{1}{x} \right) 2x - \left( \frac{1}{2} \right) 2x}{(x - 2)2x} = \frac{2 - x}{2x(x - 2)} \]

Negate the numerator and fraction bar, then cancel common factors.

\[ \frac{f(x) - f(2)}{x - 2} = - \frac{x - 2}{2x(x - 2)} = - \frac{x - 2}{2x(x - 2)} = -\frac{1}{2x} \]

In the original problem, we have a denominator of \( x - 2 \), so \( x = 2 \) is a restriction. If the body of our work, there is a fraction \( 1/x \), which is undefined when \( x = 0 \), so \( x = 0 \) is also a restriction. The remaining denominators provide no other restrictions. Hence, for all values of \( x \) except 0 and 2, the left-hand side of

\[ \frac{f(x) - f(2)}{x - 2} = -\frac{1}{2x} \]

is identical to the right-hand side.
Let’s look at another example involving function notation.

**Example 10.** Given

\[ f(x) = \frac{1}{x^2}, \]

simplify the expression

\[ \frac{f(x + h) - f(x)}{h}. \]

(11)

List all restrictions.

The function notation \( f(x + h) \) is asking us to replace each instance of \( x \) in the formula \( 1/x^2 \) with \( x + h \). Thus, \( f(x + h) = 1/(x + h)^2 \).

Here is another way to think of this substitution. Suppose that we remove the \( x \) from

\[ f(x) = \frac{1}{x^2}, \]

so that it reads

\[ f(\ ) = \frac{1}{(\ )^2}. \]

(12)

Now, if you want to compute \( f(2) \), simply insert a 2 in the blank area between parentheses. In our case, we want to compute \( f(x + h) \), so we insert an \( x + h \) in the blank space between parentheses in (12) to get

\[ f(x + h) = \frac{1}{(x + h)^2}. \]

With these preliminary remarks in mind, let’s return to the problem. First, we interpret the function notation as in our preliminary remarks and write

\[ \frac{f(x + h) - f(x)}{h} = \frac{1}{(x + h)^2} - \frac{1}{x^2}. \]

The common denominator for the numerator is found by listing each factor to the highest power that it occurs. Hence, the common denominator is \( x^2(x + h)^2 \). The denominator has no fractions to be cleared, so it suffices to multiply both numerator and denominator by \( x^2(x + h)^2 \).

\[
\frac{f(x + h) - f(x)}{h} = \frac{\left( \frac{1}{(x + h)^2} - \frac{1}{x^2} \right) x^2(x + h)^2}{h x^2(x + h)^2}
= \frac{\left( \frac{1}{(x + h)^2} \right) x^2(x + h)^2 - \left( \frac{1}{x^2} \right) x^2(x + h)^2}{h x^2(x + h)^2}
= \frac{x^2 - (x + h)^2}{h x^2(x + h)^2}
\]
We will now expand the numerator. Don’t forget to use parentheses and distribute that minus sign.

\[
\frac{f(x+h) - f(x)}{h} = \frac{x^2 - (x^2 + 2xh + h^2)}{hx^2(x+h)^2}
\]

\[
= \frac{x^2 - x^2 - 2xh - h^2}{hx^2(x+h)^2}
\]

\[
= \frac{-2xh - h^2}{hx^2(x+h)^2}
\]

Finally, factor a \(-h\) out of the numerator in hopes of finding a common factor to cancel.

\[
\frac{f(x+h) - f(x)}{h} = \frac{-h(2x + h)}{hx^2(x+h)^2}
\]

\[
= \frac{-h(2x + h)}{hx^2(x+h)^2}
\]

\[
= \frac{-(2x + h)}{x^2(x+h)^2}
\]

We must now discuss the restrictions. In the original question (11), the \(h\) in the denominator must not equal zero. Hence, \(h = 0\) is a restriction. In the final simplified form, the factor of \(x^2\) in the denominator is undefined if \(x = 0\). Hence, \(x = 0\) is a restriction. Finally, the factor of \((x + h)^2\) in the final denominator is undefined if \(x + h = 0\), so \(x = -h\) is a restriction. The remaining denominators provide no additional restrictions. Hence, provided \(h \neq 0\), \(x \neq 0\), and \(x \neq -h\), for all other combinations of \(x\) and \(h\), the left-hand side of

\[
\frac{f(x+h) - f(x)}{h} = \frac{-(2x + h)}{x^2(x+h)^2}
\]

is identical to the right-hand side.

Let’s look at one final example using function notation.

**Example 13.** If

\[
f(x) = \frac{x}{x + 1}
\]

(14)

simplify \(f(f(x))\).

We first evaluate \(f\) at \(x\), then evaluate \(f\) at the result of the first computation. Thus, we work the inner function first to obtain

\[
f(f(x)) = f\left(\frac{x}{x + 1}\right).
\]
The notation \( f(x/(x + 1)) \) is asking us to replace each occurrence of \( x \) in the formula \( x/(x + 1) \) with the expression \( x/(x + 1) \). Confusing? Here is an easy way to think of this substitution. Suppose that we remove \( x \) from
\[
f(x) = \frac{x}{x + 1},
\]
replacing each occurrence of \( x \) with empty parentheses, which will produce the template
\[
f(\ ) = \left( \frac{\ )}{\ ) + 1} \right). \tag{15}
\]
Now, if asked to compute \( f(3) \), simply insert 3 into the blank areas between parentheses. In this case, we want to compute \( f(x/(x + 1)) \), so we insert \( x/(x + 1) \) in the blank space between each set of parentheses in (15) to obtain
\[
f\left( \frac{x}{x + 1} \right) = \frac{x}{x + 1} \cdot \frac{x}{x + 1} + 1.
\]
We now have a complex fraction. The common denominator for both top and bottom of this complex fraction is \( x + 1 \). Thus, we multiply both numerator and denominator of our complex fraction by \( x + 1 \) and use the distributive property as follows.
\[
\frac{x}{x + 1} \cdot \frac{x}{x + 1} + 1 = \frac{x}{x + 1} \cdot (x + 1) + (x + 1)(x + 1)
\]
Cancel and simplify.
\[
\left( \frac{x}{x + 1} \right)(x + 1) = \frac{x}{x + 1} \cdot \frac{x}{x + 1} + (1)(x + 1)
\]
In the final denominator, the value \( x = -1/2 \) makes the denominator \( 2x + 1 \) equal to zero. Hence, \( x = -1/2 \) is a restriction. In the body of our work, several fractions have denominators of \( x + 1 \) and are therefore undefined at \( x = -1 \). Thus, \( x = -1 \) is a restriction. No other denominators add additional restrictions.
Hence, for all values of \( x \), except \( x = -1/2 \) and \( x = -1 \), the left-hand side of
\[
f(f(x)) = \frac{x}{2x + 1}
\]
is identical to the right-hand side.
In **Exercises 7.6**, evaluate the function at the given rational number. Then use the first or second technique for simplifying complex fractions explained in the narrative to simplify your answer.

1. Given
   \[ f(x) = \frac{x + 1}{2 - x}, \]
evaluate and simplify \( f(1/2) \).

2. Given
   \[ f(x) = \frac{2 - x}{x + 5}, \]
evaluate and simplify \( f(3/2) \).

3. Given
   \[ f(x) = \frac{2x + 3}{4 - x}, \]
evaluate and simplify \( f(1/3) \).

4. Given
   \[ f(x) = \frac{3 - 2x}{x + 5}, \]
evaluate and simplify \( f(2/5) \).

5. Given
   \[ f(x) = \frac{5 - 2x}{x + 4}, \]
evaluate and simplify \( f(3/5) \).

6. Given
   \[ f(x) = \frac{2x - 9}{11 - x}, \]
evaluate and simplify \( f(4/3) \).

In **Exercises 7-46**, simplify the given complex rational expression. State all restrictions.

7. \[ \frac{5 + 6}{x} = \frac{25}{x} - \frac{36}{x^3} \]

8. \[ \frac{7 + \frac{9}{x}}{49} = \frac{81}{x} - \frac{1}{x^3} \]

9. \[ \frac{7 - \frac{5}{x}}{x - 2} = \frac{8}{x - 7} + \frac{3}{x + 8} \]

10. \[ \frac{9 - \frac{7}{x}}{x + 4} = \frac{9}{x - 9} + \frac{5}{x - 4} \]

11. \[ \frac{3 + \frac{7}{x}}{9} = \frac{49}{x^2} - \frac{1}{x^3} \]
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12. \[
\frac{2 - \frac{5}{x}}{\frac{4 \cdot 25}{x^2 \cdot x^4}}
\]

13. \[
\frac{\frac{9}{x + 4} + \frac{7}{x + 9}}{\frac{9}{x + 9} + 2}
\]

14. \[
\frac{\frac{4}{x - 6} + \frac{9}{x - 9}}{\frac{9}{x - 6} + \frac{8}{x - 9}}
\]

15. \[
\frac{\frac{5}{x - 7} - \frac{4}{x - 4}}{\frac{10}{x - 4} - \frac{5}{x + 2}}
\]

16. \[
\frac{\frac{3}{x + 6} + \frac{7}{x + 9}}{\frac{9}{x + 6} - \frac{4}{x + 9}}
\]

17. \[
\frac{\frac{6}{x - 3} + \frac{5}{x - 8}}{\frac{9}{x - 3} + \frac{7}{x - 8}}
\]

18. \[
\frac{\frac{7}{x - 7} - \frac{4}{x - 2}}{\frac{7}{x - 7} - \frac{6}{x - 2}}
\]

19. \[
\frac{\frac{4}{x - 2} + \frac{7}{x - 7}}{\frac{5}{x - 2} + \frac{2}{x - 7}}
\]

20. \[
\frac{\frac{9}{x + 2} - \frac{7}{x + 5}}{\frac{4}{x + 2} + \frac{3}{x + 5}}
\]

21. \[
\frac{\frac{5 + \frac{4}{x}}{25}}{\frac{16}{x - x^3}}
\]

22. \[
\frac{\frac{6}{x + 5} + \frac{5}{x + 4}}{\frac{8}{x + 5} - \frac{3}{x + 4}}
\]

23. \[
\frac{\frac{9}{x - 5} + \frac{8}{x + 4}}{\frac{5}{x - 5} - \frac{4}{x + 4}}
\]

24. \[
\frac{\frac{4}{x - 6} + \frac{4}{x - 9}}{\frac{6}{x - 6} + \frac{6}{x - 9}}
\]

25. \[
\frac{\frac{6}{x + 8} + \frac{5}{x - 2}}{\frac{5}{x - 2} - \frac{2}{x + 2}}
\]
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<td>27.</td>
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<td>( \frac{8}{x+7} - \frac{3}{x+4} )</td>
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<td>( \frac{25 - \frac{16}{x^2}}{5 + \frac{4}{x}} )</td>
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<td>( \frac{4}{x+2} + \frac{5}{x-6} )</td>
<td>( \frac{7}{x-6} - \frac{5}{x+7} )</td>
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<td>31.</td>
<td>( \frac{2}{x-6} - \frac{4}{x+9} )</td>
<td>( \frac{3}{x^2 - 5x - 14} + \frac{2}{x^2 - 7x - 18} )</td>
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<td>32.</td>
<td>( \frac{3}{x+6} - \frac{4}{x+4} )</td>
<td>( \frac{5}{x^2 + 8x + 7} + \frac{5}{x^2 + 13x + 42} )</td>
</tr>
<tr>
<td>33.</td>
<td>( \frac{9}{x^2} - \frac{\frac{64}{x^4}}{3 - \frac{8}{x}} )</td>
<td>( \frac{4}{x-4} - \frac{8}{x-7} )</td>
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<td>( \frac{9}{x^2} - \frac{\frac{25}{x^4}}{3 - \frac{5}{x}} )</td>
<td>( \frac{4}{x-7} + \frac{2}{x+2} )</td>
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<tr>
<td>35.</td>
<td>( \frac{4}{x-4} - \frac{8}{x-7} )</td>
<td>( \frac{4}{x-7} + \frac{2}{x+2} )</td>
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<tr>
<td>36.</td>
<td>( \frac{3}{x^2 + 8x - 9} + \frac{3}{x^2 - 81} )</td>
<td>( \frac{9}{x^2 - 81} + \frac{9}{x^2 - 8x - 9} )</td>
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<tr>
<td>37.</td>
<td>( \frac{3}{x^2 + 8x - 9} + \frac{3}{x^2 - 81} )</td>
<td>( \frac{9}{x^2 - 81} + \frac{9}{x^2 - 8x - 9} )</td>
</tr>
<tr>
<td>38.</td>
<td>( \frac{7}{x^2 - 5x - 14} + \frac{2}{x^2 - 7x - 18} )</td>
<td>( \frac{5}{x^2 - 7x - 18} + \frac{8}{x^2 - 6x - 27} )</td>
</tr>
<tr>
<td>39.</td>
<td>( \frac{2}{x^2 + 8x + 7} + \frac{5}{x^2 + 13x + 42} )</td>
<td>( \frac{7}{x^2 + 13x + 42} + \frac{6}{x^2 + 3x - 18} )</td>
</tr>
</tbody>
</table>
40. \[
\frac{3}{x^2 + 5x - 14} + \frac{3}{x^2 - 5x + 14} = \frac{3}{x^2 - 7x - 98} + \frac{3}{x^2 - 15x + 14}
\]

41. \[
\frac{6}{x^2 + 11x + 24} - \frac{6}{x^2 + 13x + 40} = \frac{6}{x^2 + 13x + 40} - \frac{9}{x^2 - 3x - 40}
\]

42. \[
\frac{7}{x^2 + 13x + 30} + \frac{7}{x^2 + 19x + 90} = \frac{7}{x^2 + 19x + 90} + \frac{9}{x^2 + 19x + 90} - \frac{9}{x^2 - 7x - 18}
\]

43. \[
\frac{7}{x^2 - 6x + 5} + \frac{7}{x^2 + 2x - 35} = \frac{8}{x^2 + 2x - 35} + \frac{8}{x^2 + 8x + 7}
\]

44. \[
\frac{2}{x^2 - 4x - 12} - \frac{2}{x^2 - x - 30} = \frac{2}{x^2 - x - 30} - \frac{2}{x^2 - 4x - 45}
\]

45. \[
\frac{4}{x^2 + 6x - 7} - \frac{4}{x^2 + 2x - 3} = \frac{4}{x^2 + 2x - 3} - \frac{4}{x^2 + 5x + 6}
\]

46. \[
\frac{9}{x^2 + 3x - 4} + \frac{8}{x^2 - 7x + 6} = \frac{4}{x^2 - 7x + 6} + \frac{9}{x^2 - 10x + 24}
\]

47. Given \(f(x) = 2/x\), simplify \(f(x) - f(3)/x - 3\). State all restrictions.

48. Given \(f(x) = 5/x\), simplify \(f(x) - f(2)/x - 2\). State all restrictions.

49. Given \(f(x) = 3/x^2\), simplify \(f(x) - f(1)/x - 1\). State all restrictions.

50. Given \(f(x) = 5/x^2\), simplify \(f(x) - f(2)/x - 2\). State all restrictions.

51. Given \(f(x) = 7/x\), simplify \(f(x + h) - f(x)/h\). State all restrictions.

52. Given \(f(x) = 4/x\), simplify \(f(x + h) - f(x)/h\). State all restrictions.

53. Given \(f(x) = x + 1/3 - x\), find and simplify \(f(1/x)\). State all restrictions.
54. Given

\[ f(x) = \frac{2 - x}{3x + 4}, \]

find and simplify \( f(2/x) \). State all restrictions.

55. Given

\[ f(x) = \frac{x + 1}{2 - 5x}, \]

find and simplify \( f(5/x) \). State all restrictions.

56. Given

\[ f(x) = \frac{2x - 3}{4 + x}, \]

find and simplify \( f(1/x) \). State all restrictions.

57. Given

\[ f(x) = \frac{x}{x + 2}, \]

find and simplify \( f(f(x)) \). State all restrictions.

58. Given

\[ f(x) = \frac{2x}{x + 5}, \]

find and simplify \( f(f(x)) \). State all restrictions.
7.6 Answers

1. 1

3. 1

5. $\frac{19}{23}$

7. Provided $x \neq 0$, $-6/5$, or $6/5$,
   \[ x^2 \quad \frac{5x - 6}{5x - 6}. \]

9. Provided $x \neq 2$, $7$, $-8$, of $-43/11$,
   \[ \frac{(2x - 39)(x + 8)}{(11x + 43)(x - 2)}. \]

11. Provided $x \neq 0$, $-7/3$, or $7/3$,
    \[ x^3 \quad \frac{3x}{3x - 7}. \]

13. Provided $x \neq -4$, $-9$, $8$, or $54/11$,
    \[ \frac{(16x + 109)(x - 8)}{(11x - 54)(x + 4)}. \]

15. Provided $x \neq 7$, $4$, $-2$, or $-8$,
    \[ \frac{x + 2}{5(x - 7)}. \]

17. Provided $x \neq 3$, $8$, or $93/16$,
    \[ \frac{11x - 63}{16x - 93}. \]

19. Provided $x \neq 2$, $7$, or $39/7$,
    \[ \frac{11x - 42}{7x - 39}. \]

21. Provided $x \neq 0$, $-4/5$, or $4/5$,
    \[ \frac{x^2}{5x - 4}. \]

23. Provided $x \neq 5$, $-4$, or $-40$,
    \[ \frac{17x - 4}{x + 40}. \]

25. Provided $x \neq -8$, $2$, $-2$, or $-14/3$,
    \[ \frac{(11x + 28)(x + 2)}{(3x + 14)(x + 8)}. \]

27. Provided $x \neq -7$, $-4$, or $-11/5$,
    \[ \frac{2x - 7}{5x + 11}. \]

29. Provided $x \neq 0$ or $5/8$,
    \[ \frac{8x + 5}{x^2}. \]

31. Provided $x \neq 6$, $-9$, or $21$,
    \[ \frac{2}{3}. \]

33. Provided $x \neq 0$ or $8/3$,
    \[ \frac{3x + 8}{x^3}. \]

35. Provided $x \neq 4$, $7$, $-2$, or $1$,
    \[ \frac{-2(x + 2)}{3(x - 4)}. \]

37. Provided $x \neq 1$, $-9$, $9$, $-1$, $-5$,
    \[ \frac{(x - 5)(x + 1)}{3(x + 5)(x - 1)}. \]

39. Provided $x \neq -1$, $-7$, $-6$, $3$, $-21/13$,
    \[ \frac{(7x + 17)(x - 3)}{(13x + 21)(x + 1)}. \]
41. Provided $x \neq -3, -8, -5, 8,$
\[
-\frac{1(x - 8)}{12(x + 3)}
\]

43. Provided $x \neq 1, 5, -7, -1, 2,$
\[
\frac{7(x + 3)(x + 1)}{8(x - 2)(x - 1)}
\]

45. Provided $x \neq -7, 1, -3, -2,$
\[
\frac{-4(x + 2)}{3(x + 7)}
\]

47. Provided $x \neq 0, 3,$
\[
\frac{-2}{3x}
\]

49. Provided $x \neq 0, 1,$
\[
\frac{-3(x + 1)}{x^2}
\]

51. Provided $x \neq 0, -h,$ and $h \neq 0,$
\[
\frac{-7}{h(x + h)}
\]

53. Provided $x \neq 0, 1/3,$
\[
\frac{x + 1}{3x - 1}
\]

55. Provided $x \neq 0, 25/2,$
\[
\frac{x + 5}{2x - 25}
\]

57. Provided $x \neq -2, -4/3,$
\[
\frac{x}{3x + 4}
\]
7.7 Solving Rational Equations

When simplifying complex fractions in the previous section, we saw that multiplying both numerator and denominator by the appropriate expression could “clear” all fractions from the numerator and denominator, greatly simplifying the rational expression.

In this section, a similar technique is used.

Clear the Fractions from a Rational Equation. If your equation has rational expressions, multiply both sides of the equation by the least common denominator to clear the equation of rational expressions.

Let’s look at an example.

Example 1. Solve the following equation for $x$.

\[
\frac{x}{2} - \frac{2}{3} = \frac{3}{4}
\]

To clear this equation of fractions, we will multiply both sides by the common denominator for 2, 3, and 4, which is 12. Distribute 12 in the second step.

\[
12 \left( \frac{x}{2} - \frac{2}{3} \right) = \left( \frac{3}{4} \right) 12
\]

\[
12 \left( \frac{x}{2} \right) - 12 \left( \frac{2}{3} \right) = \left( \frac{3}{4} \right) 12
\]

Multiply.

\[
6x - 8 = 9
\]

We’ve succeeded in clearing the rational expressions from the equation by multiplying through by the common denominator. We now have a simple linear equation which can be solved by first adding 8 to both sides of the equation, followed by dividing both sides of the equation by 6.

\[
6x = 17
\]

\[
x = \frac{17}{6}
\]

We’ll leave it to our readers to check this solution.
Let’s try another example.

**Example 3.** Solve the following equation for \( x \).

\[
6 = \frac{5}{x} + \frac{6}{x^2}
\]  \hspace{1cm} (4)

In this equation, the denominators are 1, \( x \), and \( x^2 \), and the common denominator for both sides of the equation is \( x^2 \). Consequently, we begin the solution by first multiplying both sides of the equation by \( x^2 \).

\[
x^2 (6) = \left( \frac{5}{x} + \frac{6}{x^2} \right) x^2
\]

\[
x^2 (6) = \left( \frac{5}{x} \right) x^2 + \left( \frac{6}{x^2} \right) x^2
\]

Simplify.

\[
6x^2 = 5x + 6
\]

Note that multiplying both sides of the original equation by the least common denominator clears the equation of all rational expressions. This last equation is non-linear,\(^{19}\) so make one side of the equation equal to zero by subtracting \( 5x \) and 6 from both sides of the equation.

\[
6x^2 - 5x - 6 = 0
\]

To factor the left-hand side of this equation, note that it is a quadratic trinomial with \( ac = (6)(-6) = -36 \). The integer pair 4 and -9 have product -36 and sum -5. Split the middle term using this pair and factor by grouping.

\[
6x^2 + 4x - 9x - 6 = 0
\]

\[
2x(3x + 2) - 3(3x + 2) = 0
\]

\[
(2x - 3)(3x + 2) = 0
\]

The zero product property forces either

\[
2x - 3 = 0 \quad \text{or} \quad 3x + 2 = 0.
\]

Each of these linear equations is easily solved.

\[
x = \frac{3}{2} \quad \text{or} \quad x = -\frac{2}{3}
\]

Of course, we should always check our solutions. Substituting \( x = 3/2 \) into the right-hand side of the original equation (4),

\(^{19}\) Whenever an equation in \( x \) has a power of \( x \) other than 1, the equation is nonlinear (the graphs involved are not all lines). As we’ve seen in previous chapters, the approach to solving a quadratic (second degree) equation should be to make one side of the equation equal to zero, then factor or use the quadratic formula to find the solutions.

Version: Fall 2007
\[
\frac{5}{x} + \frac{6}{x^2} = \frac{5}{3/2} + \frac{6}{(3/2)^2} = \frac{5}{3/2} + \frac{6}{9/4}.
\]

In the final expression, multiply top and bottom of the first fraction by 2, top and bottom of the second fraction by 4.

\[
\frac{5}{3/2} \cdot \frac{2}{2} + \frac{6}{9/4} \cdot \frac{4}{4} = \frac{10}{3} + \frac{24}{9}
\]

Make equivalent fractions with a common denominator of 9 and add.

\[
\frac{10}{3} \cdot \frac{3}{3} + \frac{24}{9} = \frac{30}{9} + \frac{24}{9} = \frac{54}{9} = 6
\]

Note that this result is identical to the left-hand side of the original equation \(4\). Thus, \(x = 3/2\) checks.

This example clearly demonstrates that the check can be as difficult and as time consuming as the computation used to originally solve the equation. For this reason, we tend to get lazy and not check our answers as we should. There is help, however, as the graphing calculator can help us check the solutions of equations.

First, enter the solution \(3/2\) in your calculator screen, push the \(\text{STO}\) button, then push the \(\times\) button, and execute the resulting command on the screen by pushing the \(\text{ENTER}\) key. The result is shown in Figure 1(a).

Next, enter the expression \(5/x+6/x^2\) and execute the resulting command on the screen by pushing the \(\text{ENTER}\) key. The result is shown in Figure 1(b). Note that the result is 6, the same as computed by hand above, and it matches the left-hand side of the original equation \(4\). We’ve also used the calculator to check the second solution \(x = -2/3\). This is shown in Figure 4(c).

![Figure 1](a) (b) (c)

**Figure 1.** Using the graphing calculator to check the solutions of equation \(4\).

Let’s look at another example.

**Example 5.** Solve the following equation for \(x\).

\[
\frac{2}{x^2} = 1 - \frac{2}{x}
\]

First, multiply both sides of equation \(6\) by the common denominator \(x^2\).
\[
x^2 \left( \frac{2}{x^2} \right) = \left( 1 - \frac{2}{x} \right) x^2
\]
\[
2 = x^2 - 2x
\]

Make one side zero.

\[
0 = x^2 - 2x - 2
\]

The right-hand side is a quadratic trinomial with \(ac = (1)(-2) = -2\). There are no integer pairs with product \(-2\) that sum to \(-2\), so this quadratic trinomial does not factor. Fortunately, the equation is quadratic (second degree), so we can use the quadratic formula with \(a = 1\), \(b = -2\), and \(c = -2\).

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} = \frac{2 \pm \sqrt{12}}{2}
\]

This gives us two solutions, \(x = (2 - \sqrt{12})/2\) and \(x = (2 + \sqrt{12})/2\). Let’s check the solution \(x = (2 - \sqrt{12})/2\). First, enter this result in your calculator, press the \texttt{STO} \texttt{I} button, press \(x\), then press the \texttt{ENTER} key to execute the command and store the solution in the variable \(x\). This command is shown in Figure 2(a).

Enter the left-hand side of the original equation (6) as \(2/x^2\) and press the \texttt{ENTER} key to execute this command. This is shown in Figure 2(b).

Enter the right-hand side of the original equation (6) as \(1 - 2/x\) and press the \texttt{ENTER} key to execute this command. This is shown in Figure 2(c). Note that the left- and right-hand sides of equation (6) are both shown to equal 3.732050808 at \(x = (2 - \sqrt{12})/2\) (at \(x = -0.7320508076\)), as shown in Figure 2(c). This shows that \(x = (2 - \sqrt{12})/2\) is a solution of equation (6).

We leave it to our readers to check the second solution, \(x = (2 + \sqrt{12})/2\).

\begin{figure}
\centering
\begin{subfigure}{0.3\textwidth}
\includegraphics[width=\textwidth]{figure2a.png}
\caption{(a)}
\end{subfigure} \quad
\begin{subfigure}{0.3\textwidth}
\includegraphics[width=\textwidth]{figure2b.png}
\caption{(b)}
\end{subfigure} \quad
\begin{subfigure}{0.3\textwidth}
\includegraphics[width=\textwidth]{figure2c.png}
\caption{(c)}
\end{subfigure}
\caption{Using the graphing calculator to check the solutions of equation (6).}
\end{figure}

Let’s look at another example, this one involving function notation.

**Example 7.** Consider the function defined by

\[
f(x) = \frac{1}{x} + \frac{1}{x-4}.
\]
Solve the equation \( f(x) = 2 \) for \( x \) using both graphical and analytical techniques, then compare solutions. Perform each of the following tasks.

a. Sketch the graph of \( f \) on graph paper. Label the zeros of \( f \) with their coordinates and the asymptotes of \( f \) with their equations.

b. Add the graph of \( y = 2 \) to your plot and estimate the coordinates of where the graph of \( f \) intersects the graph of \( y = 2 \).

c. Use the \textit{intersect} utility on your calculator to find better approximations of the points where the graphs of \( f \) and \( y = 2 \) intersect.

d. Solve the equation \( f(x) = 2 \) algebraically and compare your solutions to those found in part (c).

For the graph in part (a), we need to find the zeros of \( f \) and the equations of any vertical or horizontal asymptotes.

To find the zero of the function \( f \), we find a common denominator and add the two rational expressions in \textit{equation (8)}.

\[
f(x) = \frac{1}{x} + \frac{1}{x - 4} = \frac{x - 4}{x(x - 4)} + \frac{x}{x(x - 4)} = \frac{2x - 4}{x(x - 4)} \quad (9)
\]

Note that the numerator of this result equal zero (but not the denominator) when \( x = 2 \). This is the zero of \( f \). Thus, the graph of \( f \) has \( x \)-intercept at \( (2, 0) \), as shown in \textit{Figure 4}.

Note that the rational function in \textit{equation (9)} is reduced to lowest terms. The denominators of \( x \) and \( x + 4 \) in \textit{equation (9)} are zero when \( x = 0 \) and \( x = 4 \). These are our vertical asymptotes, as shown in \textit{Figure 4}.

To find the horizontal asymptotes, we need to examine what happens to the function values as \( x \) increases (or decreases) without bound. Enter the function in the \( Y= \) menu with \( 1/X+1/(X-4) \), as shown in \textit{Figure 3}(a). Press 2nd \texttt{TBLSET}, then highlight \texttt{ASK} for the independent variable and press \texttt{ENTER} to make this selection permanent, as shown in \textit{Figure 3}(b).

Press 2nd \texttt{TABLE}, then enter 10, 100, 1,000, and 10,000, as shown in \textit{Figure 3}(c). Note how the values of \( Y1 \) approach zero. In \textit{Figure 3}(d), as \( x \) decreases without bound, the end-behavior is the same. This is an indication of a horizontal asymptote at \( y = 0 \), as shown in \textit{Figure 4}.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Examining the end-behavior of \( f \) with the graphing calculator.}
\end{figure}
Chapter 7 Rational Functions

At this point, we already have our function $f$ loaded in Y1, so we can press the ZOOM button and select 6:ZStandard to produce the graph shown in Figure 5. As expected, the graphing calculator does not do a very good job with the rational function $f$, particularly near the discontinuities at the vertical asymptotes. However, there is enough information in Figure 4, to draw a very nice graph of the rational function on our graph paper, as shown in Figure 6(a). Note: We haven’t labeled asymptotes with equations, nor zeros with coordinates, in Figure 6(a), as we thought the picture might be a little crowded. However, you should label each of these parts on your graph paper, as we did in Figure 4.

![Figure 4](image)

**Figure 4.** Placing the horizontal and vertical asymptotes and the $x$-intercept of the graph of the function $f$.

![Figure 5](image)

**Figure 5.** The graph of $f$ as drawn on the calculator.

Let’s now address part (b) by adding the horizontal line $y = 2$ to the graph, as shown in Figure 6(b). Note that the graph of $y = 2$ intersects the graph of the rational function $f$ at two points $A$ and $B$. The $x$-values of points $A$ and $B$ are the solutions to our equation $f(x) = 2$.

We can get a crude estimate of the $x$-coordinates of points $A$ and $B$ right off our graph paper. The $x$-value of point $A$ is approximately $x \approx 0.3$, while the $x$-value of point $B$ appears to be approximately $x \approx 4.6$. 

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Section 7.7 Solving Rational Equations

Next, let’s address the task required in part (c). We have very reasonable estimates of the solutions of \( f(x) = 2 \) based on the data presented in Figure 6(b). Let’s use the graphing calculator to improve upon these estimates.

First, load the equation \( Y_2 = 2 \) into the \( Y= \) menu, as shown in Figure 7(a). We need to find where the graph of \( Y_1 \) intersects the graph of \( Y_2 \), so we press 2nd CALC and select 5:intersect from the menu. In the usual manner, select “First curve,” “Second curve,” and move the cursor close to the point you wish to estimate. This is your “Guess.” Perform similar tasks for the second point of intersection.

Our results are shown in Figures 7(b) and Figures 7(c). The estimate in Figure 7(b) has \( x \approx 0.43844719 \), while that in Figure 7(c) has \( x \approx 4.5615528 \). Note that these are more accurate than the approximations of \( x \approx 0.3 \) and \( x \approx 4.6 \) captured from our hand drawn image in Figure 6(b).

Finally, let’s address the request for an algebraic solution of \( f(x) = 2 \) in part (d). First, replace \( f(x) \) with \( 1/x + 1/(x-4) \) to obtain

\[
\frac{1}{x} + \frac{1}{x-4} = 2.
\]

Multiply both sides of this equation by the common denominator \( x(x-4) \).
\[ x(x - 4) \left( \frac{1}{x} + \frac{1}{x - 4} \right) = [2] x(x - 4) \]

\[ x(x - 4) \left( \frac{1}{x} \right) + x(x - 4) \left( \frac{1}{x - 4} \right) = [2] x(x - 4) \]

Cancel.

\[ x(x - 4) \left( \frac{1}{x} \right) + x(x - 4) \left( \frac{1}{x - 4} \right) = [2] x(x - 4) \]

Simplify each side.

\[ 2x - 4 = 2x^2 - 8x \]

This last equation is nonlinear, so we make one side zero by subtracting 2x and adding 4 to both sides of the equation.

\[ 0 = 2x^2 - 8x - 2x + 4 \]
\[ 0 = 2x^2 - 10x + 4 \]

Note that each coefficient on the right-hand side of this last equation is divisible by 2. Let’s divide both sides of the equation by 2, distributing the division through each term on the right-hand side of the equation.

\[ 0 = x^2 - 5x + 2 \]

The trinomial on the right is a quadratic with \( ac = (1)(2) = 2 \). There are no integer pairs having product 2 and sum \(-5\), so this trinomial doesn’t factor. We will use the quadratic formula instead, with \( a = 1 \), \( b = -5 \) and \( c = 2 \).

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(2)}}{2(1)} = \frac{5 \pm \sqrt{17}}{2} \]

It remains to compare these with the graphical solutions found in part (c). So, enter the solution \((5 - \sqrt{17})/2\) in your calculator screen, as shown in Figure 8(a). Enter \((5 + \sqrt{17})/2\), as shown in Figure 8(b). Thus,

\[ \frac{5 - \sqrt{17}}{2} \approx 0.4384471872 \quad \text{and} \quad \frac{5 + \sqrt{17}}{2} \approx 4.561552813 \]

Note the close agreement with the approximations found in part (c).

Figure 8. Approximating the exact solutions.
Let’s look at another example.

**Example 10.** Solve the following equation for \( x \), both graphically and analytically.

\[
\frac{1}{x+2} - \frac{x}{2-x} = \frac{x+6}{x^2-4} \tag{11}
\]

We start the graphical solution in the usual manner, loading the left- and right-hand sides of **equation (11)** into \( Y_1 \) and \( Y_2 \), as shown in **Figure 9(a)**. Note that in the resulting plot, shown in **Figure 9(b)**, it is very difficult to interpret where the graph of the left-hand side intersects the graph of the right-hand side of **equation (11)**.

![Figure 9](image)

**Figure 9.** Sketch the left- and right-hand sides of **equation (11)**.

In this situation, a better strategy is to make one side of **equation (11)** equal to zero.

\[
\frac{1}{x+2} - \frac{x}{2-x} - \frac{x+6}{x^2-4} = 0 \tag{12}
\]

Our approach will now change. We’ll plot the left-hand side of **equation (12)**, then find where the left-hand side is equal to zero; that is, we’ll find where the graph of the left-hand side of **equation (12)** intercepts the \( x \)-axis.

With this thought in mind, load the left-hand side of **equation (12)** into \( Y_1 \), as shown in **Figure 10(a)**. Note that the graph in **Figure 10(b)** appears to have only one vertical asymptote at \( x = -2 \) (some cancellation must remove the factor of \( x-2 \) from the denominator when you combine the terms of the left-hand side of **equation (12)**). Further, when you use the **zero** utility in the **CALC** menu of the graphing calculator, there appears to be a zero at \( x = -4 \), as shown in **Figure 10(b)**.

![Figure 10](image)

**Figure 10.** Finding the zero of the left-hand side of **equation (12)**.

---

20 Closer analysis might reveal a “hole” in the graph, but we push on because our check at the end of the problem will reveal a false solution.
Therefore, equation (12) seems to have only one solution, namely $x = 4$.

Next, let’s seek an analytical solution of equation (11). We’ll need to factor the denominators in order to discover a common denominator.

$$\frac{1}{x+2} - \frac{x}{2-x} = \frac{x+6}{(x+2)(x-2)}$$

It’s tempting to use a denominator of $(x+2)(2-x)(x-2)$. However, the denominator of the second term on the left-hand side of this last equation, $2-x$, is in a different order than the factors in the other denominators, $x-2$ and $x+2$, so let’s perform a sign change on this term and reverse the order. We will negate the fraction bar and negate the denominator. That’s two sign changes, so the term remains unchanged when we write

$$\frac{1}{x+2} + \frac{x}{x-2} = \frac{x+6}{(x+2)(x-2)}.$$

Now we see that a common denominator of $(x+2)(x-2)$ will suffice. Let’s multiply both sides of the last equation by $(x+2)(x-2)$.

$$(x+2)(x-2) \left[ \frac{1}{x+2} + \frac{x}{x-2} \right] = \left[ \frac{x+6}{(x+2)(x-2)} \right] (x+2)(x-2)$$

Cancel.

$$(x+2)(x-2) \left[ \frac{1}{x+2} \right] + (x+2)(x-2) \left[ \frac{x}{x-2} \right] = \left[ \frac{x+6}{(x+2)(x-2)} \right] (x+2)(x-2)$$

Simplify.

$$x - 2 + x^2 + 2x = x + 6$$
$$x^2 + 3x - 2 = x + 6$$

This last equation is nonlinear because of the presence of a power of $x$ larger than 1 (note the $x^2$ term). Therefore, the strategy is to make one side of the equation equal to zero. We will subtract $x$ and subtract 6 from both sides of the equation.

$$x^2 + 3x - 2 - x - 6 = 0$$
$$x^2 + 2x - 8 = 0$$

The left-hand side is a quadratic trinomial with $ac = (1)(-8) = -8$. The integer pair 4 and $-2$ have product $-8$ and sum 2. Thus,

$$(x + 4)(x - 2) = 0.$$

Using the zero product property, either

$$x + 4 = 0 \quad \text{or} \quad x - 2 = 0,$$
so

\[ x = -4 \quad \text{or} \quad x = 2. \]

The fact that we have found two answers using an analytical method is troubling. After all, the graph in Figure 10(b) indicates only one solution, namely \( x = -4 \). It is comforting that one of our analytical solutions is also \( x = -4 \), but it is still disconcerting that our analytical approach reveals a second “answer,” namely \( x = 2 \).

However, notice that we haven’t paid any attention to the restrictions caused by denominators up to this point. Indeed, careful consideration of equation (11) reveals factors of \( x + 2 \) and \( x - 2 \) in the denominators. Hence, \( x = -2 \) and \( x = 2 \) are restrictions.

Note that one of our answers, namely \( x = 2 \), is a restricted value. It will make some of the denominators in equation (11) equal to zero, so it cannot be a solution. Thus, the only viable solution is \( x = -4 \). One can certainly check this solution by hand, but let’s use the graphing calculator to assist us in the check.

First, enter \(-4\), press the STO button, press X, then press ENTER to execute the resulting command and store \(-4\) in the variable X. The result is shown in Figure 11(a).

Next, we calculate the value of the left-hand side of equation (11) at this value of X. Enter the left-hand side of equation (11) as \( 1/(X+2) - X/(2-X) \), then press the ENTER key to execute the statement and produce the result shown in Figure 11(b).

Finally, enter the right-hand side of equation (11) as \( (X+6)/(x^2-4) \) and press the ENTER key to execute the statement. The result is shown in Figure 11(c). Note that both sides of the equation equal \( 0.1666666667 \) at \( X = -4 \). Thus, the solution \( x = -4 \) checks.

\[ \begin{align*}
\text{(a)} & \quad \begin{array}{c}
-4 + X \\
\end{array} \\
\text{(b)} & \quad \begin{array}{c}
1/(X+2) - X/(2-X) \\
0.1666666667 \\
\end{array} \\
\text{(c)} & \quad \begin{array}{c}
1/(X+2) - X/(2-X) \\
+6)/(x^2-4) \\
0.1666666667 \\
\end{array}
\end{align*} \]

Figure 11. Using the graphing calculator to check the solution \( x = -4 \) of equation (11).
7.7 Exercises

For each of the rational functions given in Exercises 1-6, perform each of the following tasks.

i. Set up a coordinate system on graph paper. Label and scale each axis. Remember to draw all lines with a ruler.

ii. Plot the zero of the rational function on your coordinate system and label it with its coordinates. Plot the vertical and horizontal asymptotes on your coordinate system and label them with their equations. Use this information (and your graphing calculator) to draw the graph of \( f \).

iii. Plot the horizontal line \( y = k \) on your coordinate system and label this line with its equation.

iv. Use your calculator’s intersect utility to help determine the solution of \( f(x) = k \). Label this point on your graph with its coordinates.

v. Solve the equation \( f(x) = k \) algebraically, placing the work for this solution on your graph paper next to your coordinate system containing the graphical solution. Do the answers agree?

6. \( f(x) = \frac{5 - 2x}{x - 1}; \quad k = 3 \)

In Exercises 7-14, use a strictly algebraic technique to solve the equation \( f(x) = k \) for the given function and value of \( k \). You are encouraged to check your result with your calculator.

7. \( f(x) = \frac{16x - 9}{2x - 1}; \quad k = 8 \)

8. \( f(x) = \frac{10x - 3}{7x + 7}; \quad k = 1 \)

9. \( f(x) = \frac{5x + 8}{4x + 1}; \quad k = -11 \)

10. \( f(x) = \frac{-6x - 11}{7x - 2}; \quad k = -6 \)

11. \( f(x) = \frac{-35x}{7x + 12}; \quad k = -5 \)

12. \( f(x) = \frac{66x - 5}{6x - 10}; \quad k = -11 \)

13. \( f(x) = \frac{8x + 2}{x - 11}; \quad k = 11 \)

14. \( f(x) = \frac{36x - 7}{3x - 4}; \quad k = 12 \)

In Exercises 15-20, use a strictly algebraic technique to solve the given equation. You are encouraged to check your result with your calculator.

15. \( \frac{x}{7} + \frac{8}{9} = -\frac{8}{7} \)

16. \( \frac{x}{3} + \frac{9}{2} = -\frac{3}{8} \)

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17. \(-\frac{57}{x} = 27 \frac{40}{x^2}\)

18. \(-\frac{117}{x} = 54 + \frac{54}{x^2}\)

19. \(\frac{7}{x} = 4 - \frac{3}{x^2}\)

20. \(\frac{3}{x^2} = 5 - \frac{3}{x}\)

For each of the rational functions given in Exercises 21-26, perform each of the following tasks.

i. Set up a coordinate system on graph paper. Label and scale each axis. Remember to draw all lines with a ruler.

ii. Plot the zero of the rational function on your coordinate system and label it with its coordinates. You may use your calculator’s zero utility to find this, if you wish.

iii. Plot the vertical and horizontal asymptotes on your coordinate system and label them with their equations. Use the asymptote and zero information (and your graphing calculator) to draw the graph of \(f\).

iv. Plot the horizontal line \(y = k\) on your coordinate system and label this line with its equation.

v. Use your calculator’s intersect utility to help determine the solution of \(f(x) = k\). Label this point on your graph with its coordinates.

vi. Solve the equation \(f(x) = k\) algebraically, placing the work for this solution on your graph paper next to your coordinate system containing the graphical solution. Do the answers agree?

21. \(f(x) = \frac{1}{x} + \frac{1}{x + 5}, \quad k = 9/14\)

22. \(f(x) = \frac{1}{x} + \frac{1}{x - 2}, \quad k = 8/15\)

23. \(f(x) = \frac{1}{x - 1} - \frac{1}{x + 1}, \quad k = 1/4\)

24. \(f(x) = \frac{1}{x - 1} - \frac{1}{x + 2}, \quad k = 1/6\)

25. \(f(x) = \frac{1}{x - 2} + \frac{1}{x + 2}, \quad k = 4\)

26. \(f(x) = \frac{1}{x - 3} + \frac{1}{x + 2}, \quad k = 5\)

In Exercises 27-34, use a strictly algebraic technique to solve the given equation. You are encouraged to check your result with your calculator.

27. \(\frac{2}{x + 1} + \frac{4}{x + 2} = -3\)

28. \(\frac{2}{x - 5} - \frac{7}{x - 7} = 9\)

29. \(\frac{3}{x + 9} - \frac{2}{x + 7} = -3\)

30. \(\frac{3}{x + 9} - \frac{6}{x + 7} = 9\)

31. \(\frac{2}{x + 9} + \frac{2}{x + 6} = -1\)

32. \(\frac{5}{x - 6} - \frac{8}{x - 7} = -1\)

33. \(\frac{3}{x + 3} + \frac{6}{x + 2} = -2\)

34. \(\frac{2}{x - 4} - \frac{2}{x - 1} = 1\)
For each of the equations in Exercises 35-40, perform each of the following tasks.

i. Follow the lead of Example 10 in the text. Make one side of the equation equal to zero. Load the nonzero side into your calculator and draw its graph.

ii. Determine the vertical asymptotes of by analyzing the equation and the resulting graph on your calculator. Use the TABLE feature of your calculator to determine any horizontal asymptote behavior.

iii. Use the zero finding utility in the CALC menu to determine the zero of the nonzero side of the resulting equation.

iv. Set up a coordinate system on graph paper. Label and scale each axis. Remember to draw all lines with a ruler. Draw the graph of the nonzero side of the equation. Draw the vertical and horizontal asymptotes and label them with their equations. Plot the x-intercept and label it with its coordinates.

v. Use an algebraic technique to determine the solution of the equation and compare it with the solution found by the graphical analysis above.

35. \[ \frac{x}{x+1} + \frac{8}{x^2-2x-3} = \frac{2}{x-3} \]

36. \[ \frac{x}{x+4} - \frac{2}{x+1} = \frac{12}{x^2+5x+4} \]

37. \[ \frac{x}{x+1} - \frac{4}{2x+1} = \frac{2x-1}{2x^2+3x+2} \]

38. \[ \frac{2x}{x-4} - \frac{1}{x+1} = \frac{4x+24}{x^2-3x-4} \]

39. \[ \frac{x}{x-2} + \frac{3}{x+2} = \frac{8}{4-x^2} \]

40. \[ \frac{x}{x-1} - \frac{4}{x+1} = \frac{x-6}{1-x^2} \]

In Exercises 41-68, use a strictly algebraic technique to solve the given equation. You are encouraged to check your result with your calculator.

41. \[ \frac{x}{3x-9} - \frac{9}{x} = \frac{1}{x-3} \]

42. \[ \frac{5x}{x+2} + \frac{5}{x-5} = \frac{x+6}{x^2-3x-10} \]

43. \[ \frac{3x}{x+2} - \frac{7}{x} = -\frac{1}{2x+4} \]

44. \[ \frac{4x}{x+6} - \frac{4}{x+4} = \frac{x-4}{x^2+10x+24} \]

45. \[ \frac{x}{x-5} + \frac{9}{4-x} = \frac{x+5}{x^2-9x+20} \]

46. \[ \frac{6x}{x-5} - \frac{2}{x-3} = \frac{x-8}{x^2-8x+15} \]

47. \[ \frac{2x}{x-4} + \frac{5}{2-x} = \frac{x+8}{x^2-6x+8} \]

48. \[ \frac{x}{x-7} - \frac{8}{5-x} = \frac{x+7}{x^2-12x+35} \]

49. \[ -\frac{2}{2x+2} - \frac{6}{x} = -\frac{2}{x+1} \]

50. \[ \frac{7x}{x+3} - \frac{4}{2-x} = \frac{x+8}{x^2+x-6} \]

51. \[ \frac{2x}{x+5} - \frac{2}{6-x} = \frac{x-2}{x^2-x-30} \]

52. \[ \frac{4x}{x+1} + \frac{6}{x+3} = \frac{x-9}{x^2+4x+3} \]

53. \[ \frac{x}{x+7} - \frac{2}{x+5} = \frac{x+1}{x^2+12x+35} \]

54. \[ \frac{5x}{6x+4} + \frac{6}{x} = \frac{1}{3x+2} \]

55. \[ \frac{2x}{3x+9} - \frac{4}{x} = -\frac{2}{x+3} \]
56. \( \frac{7x}{x+1} - \frac{4}{x+2} = \frac{x+6}{x^2+3x+2} \)

57. \( \frac{x}{2x-8} + \frac{8}{x} = \frac{2}{x-4} \)

58. \( \frac{3x}{x-6} + \frac{6}{x-6} = \frac{x+2}{x^2-12x+36} \)

59. \( \frac{x}{x+2} + \frac{2}{x} = -\frac{5}{2x+4} \)

60. \( \frac{4x}{x-2} + \frac{2}{2-x} = \frac{x+4}{x^2-4x+4} \)

61. \( -\frac{2x}{3x-9} - \frac{3}{x} = -\frac{2}{x-3} \)

62. \( \frac{2x}{x+1} - \frac{2}{x} = \frac{1}{2x+2} \)

63. \( \frac{x}{x+1} + \frac{5}{x} = \frac{1}{4x+4} \)

64. \( \frac{2x}{x-4} - \frac{8}{x-7} = \frac{x+2}{x^2-11x+28} \)

65. \( -\frac{9x}{8x-2} + \frac{2}{x} = -\frac{2}{4x-1} \)

66. \( \frac{2x}{x-3} - \frac{4}{4-x} = \frac{x-9}{x^2-7x+12} \)

67. \( \frac{4x}{x+6} - \frac{5}{7-x} = \frac{x-5}{x^2-x-42} \)

68. \( \frac{x}{x-1} - \frac{4}{x} = \frac{1}{5x-5} \)
## 7.7 Answers

1. $x = -7/2$

2. $x = 5/3$

3. $x = 0$

4. none

5. $x = -1, x = 1$

6. $x = 1/4$

7. none

8. $x = -3/2, x = 3/2, x = 3$

9. $-19, 49$

10. none

11. $-128, 9$

12. $7 + \sqrt{97}, 7 - \sqrt{97}/8$

13. $-19, 49$

14. $-128, 9$

15. $7 + \sqrt{97}, 7 - \sqrt{97}/8$

16. $-19, 49$

17. $-128, 9$

18. $7 + \sqrt{97}, 7 - \sqrt{97}/8$

19. $-19, 49$

20. $-128, 9$

21. $x = -35/9$ or $x = 2$

22. $x = -1, x = 1$

23. $x = -3$ or $x = 3$
25. \( x = \frac{1 + \sqrt{65}}{4}, \frac{1 - \sqrt{65}}{4} \)

27. \( x = -\frac{15 + \sqrt{57}}{6}, -\frac{15 - \sqrt{57}}{6} \)

29. \( x = -\frac{49 + \sqrt{97}}{6}, -\frac{49 - \sqrt{97}}{6} \)

31. \(-7, -12\)

33. \( x = -\frac{19 + \sqrt{73}}{4}, -\frac{19 - \sqrt{73}}{4} \)

35. \( x = 2 \)

37. \( x = 3 \)

39. \( x = -\frac{5 + \sqrt{17}}{2}, -\frac{5 - \sqrt{17}}{2} \)

41. \( 27 \)

43. \( \frac{7}{2}, -\frac{4}{3} \)

45. 10

47. 3

49. \(-6, -2\)

51. \(4, \frac{3}{2}\)

53. 3

55. 6

57. \(-16\)
59. \(-\frac{9 + \sqrt{17}}{4}, \frac{-9 - \sqrt{17}}{4}\)

61. \(-\frac{9}{2}\)

63. \(-\frac{19 + \sqrt{41}}{8}, \frac{-19 - \sqrt{41}}{8}\)

65. \(\frac{2}{9}, 2\)

67. \(\frac{7}{2}, \frac{5}{2}\)
7.8 Applications of Rational Functions

In this section, we will investigate the use of rational functions in several applications.

Number Problems

We start by recalling the definition of the reciprocal of a number.

**Definition 1.** For any nonzero real number \( a \), the **reciprocal** of \( a \) is the number \( 1/a \). Note that the product of a number and its reciprocal is always equal to the number 1. That is,

\[
a \cdot \frac{1}{a} = 1.
\]

For example, the reciprocal of the number 3 is \( 1/3 \). Note that we simply “invert” the number 3 to obtain its reciprocal \( 1/3 \). Further, note that the product of 3 and its reciprocal \( 1/3 \) is

\[
3 \cdot \frac{1}{3} = 1.
\]

As a second example, to find the reciprocal of \(-3/5\), we could make the calculation

\[
\frac{1}{-\frac{3}{5}} = 1 \div \left(-\frac{3}{5}\right) = 1 \cdot \left(-\frac{5}{3}\right) = -\frac{5}{3},
\]

but it’s probably faster to simply “invert” \(-3/5\) to obtain its reciprocal \(-5/3\). Again, note that the product of \(-3/5\) and its reciprocal \(-5/3\) is

\[
\left(-\frac{3}{5}\right) \cdot \left(-\frac{5}{3}\right) = 1.
\]

Let’s look at some applications that involve the reciprocals of numbers.

**Example 2.** The sum of a number and its reciprocal is 29/10. Find the number(s).

Let \( x \) represent a nonzero number. The reciprocal of \( x \) is \( 1/x \). Hence, the sum of \( x \) and its reciprocal is represented by the rational expression \( x + 1/x \). Set this equal to 29/10.

\[
x + \frac{1}{x} = \frac{29}{10}
\]

To clear fractions from this equation, multiply both sides by the common denominator 10x.
\[10x \left( x + \frac{1}{x} \right) = \left( \frac{29}{10} \right) 10x\]
\[10x^2 + 10 = 29x\]

This equation is nonlinear (it has a power of \(x\) larger than 1), so make one side equal to zero by subtracting \(29x\) from both sides of the equation.
\[10x^2 - 29x + 10 = 0\]

Let’s try to use the \(ac\)-test to factor. Note that \(ac = (10)(10) = 100\). The integer pair \((-4, -25)\) has product 100 and sum \(-29\). Break up the middle term of the quadratic trinomial using this pair, then factor by grouping.
\[10x^2 - 4x - 25x + 10 = 0\]
\[2x(5x - 2) - 5(5x - 2) = 0\]
\[(2x - 5)(5x - 2) = 0\]

Using the zero product property, either
\[2x - 5 = 0 \quad \text{or} \quad 5x - 2 = 0.\]

Each of these linear equations is easily solved.
\[x = \frac{5}{2} \quad \text{or} \quad x = \frac{2}{5}\]

Hence, we have two solutions for \(x\). However, they both lead to the same number-reciprocal pair. That is, if \(x = 5/2\), then its reciprocal is \(2/5\). On the other hand, if \(x = 2/5\), then its reciprocal is \(5/2\).

Let’s check our solution by taking the sum of the solution and its reciprocal. Note that
\[\frac{5}{2} + \frac{2}{5} = \frac{25}{10} + \frac{4}{10} = \frac{29}{10},\]
as required by the problem statement.

Let’s look at another application of the reciprocal concept.

**Example 3.** There are two numbers. The second number is 1 larger than twice the first number. The sum of the reciprocals of the two numbers is \(7/10\). Find the two numbers.

Let \(x\) represent the first number. If the second number is 1 larger than twice the first number, then the second number can be represented by the expression \(2x + 1\).

Thus, our two numbers are \(x\) and \(2x + 1\). Their reciprocals, respectively, are \(1/x\) and \(1/(2x + 1)\). Therefore, the sum of their reciprocals can be represented by the rational expression \(1/x + 1/(2x + 1)\). Set this equal to \(7/10\).
\[
\frac{1}{x} + \frac{1}{2x+1} = \frac{7}{10}
\]

Multiply both sides of this equation by the common denominator \(10x(2x+1)\).

\[
10x(2x+1) \left[ \frac{1}{x} + \frac{1}{2x+1} \right] = 10x(2x+1)
\]

\[
10(2x+1) + 10x = 7x(2x+1)
\]

Expand and simplify each side of this result.

\[
20x + 10 + 10x = 14x^2 + 7x
\]

\[
30x + 10 = 14x^2 + 7x
\]

Again, this equation is nonlinear. We will move everything to the right-hand side of this equation. Subtract \(30x\) and \(10\) from both sides of the equation to obtain

\[
0 = 14x^2 + 7x - 30x - 10
\]

\[
0 = 14x^2 - 23x - 10.
\]

Note that the right-hand side of this equation is quadratic with \(ac = (14)(-10) = -140\). The integer pair \(\{5, -28\}\) has product \(-140\) and sum \(-23\). Break up the middle term using this pair and factor by grouping.

\[
0 = 14x^2 + 5x - 28x - 10
\]

\[
0 = x(14x + 5) - 2(14x + 5)
\]

\[
0 = (x - 2)(14x + 5)
\]

Using the zero product property, either

\[
x - 2 = 0 \quad \text{or} \quad 14x + 5 = 0.
\]

These linear equations are easily solved for \(x\), providing

\[
x = 2 \quad \text{or} \quad x = -\frac{5}{14}.
\]

We still need to answer the question, which was to find two numbers such that the sum of their reciprocals is \(7/10\). Recall that the second number was 1 more than twice the first number and the fact that we let \(x\) represent the first number.

Consequently, if the first number is \(x = 2\), then the second number is \(2x + 1\), or \(2(2) + 1\). That is, the second number is 5. Let’s check to see if the pair \(\{2, 5\}\) is a solution by computing the sum of the reciprocals of 2 and 5.

\[
\frac{1}{2} + \frac{1}{5} = \frac{5}{10} + \frac{2}{10} = \frac{7}{10}
\]

Thus, the pair \(\{2, 5\}\) is a solution.
However, we found a second value for the first number, namely \( x = -\frac{5}{14} \). If this is the first number, then the second number is
\[
2 \left( -\frac{5}{14} \right) + 1 = -\frac{5}{7} + \frac{7}{7} = \frac{2}{7}.
\]
Thus, we have a second pair \( \{-\frac{5}{14}, \frac{2}{7}\} \), but what is the sum of the reciprocals of these two numbers? The reciprocals are \(-\frac{14}{5} \) and \( \frac{7}{2} \), and their sum is
\[
-\frac{14}{5} + \frac{7}{2} = -\frac{28}{10} + \frac{35}{10} = \frac{7}{10},
\]
as required by the problem statement. Hence, the pair \( \{-\frac{14}{5}, \frac{7}{2}\} \) is also a solution.

**Distance, Speed, and Time Problems**

When we developed the *Equations of Motion* in the chapter on quadratic functions, we showed that if an object moves with constant speed, then the distance traveled is given by the formula
\[
d = vt,
\]
where \( d \) represents the distance traveled, \( v \) represents the speed, and \( t \) represents the time of travel.

For example, if a car travels down a highway at a constant speed of 50 miles per hour (50 mi/h) for 4 hours (4 h), then it will travel
\[
d = vt
\]
\[
d = 50 \frac{\text{mi}}{\text{h}} \times 4 \text{ h}
\]
\[
d = 200 \text{ mi}.
\]
Let’s put this relation to use in some applications.

**Example 5.** A boat travels at a constant speed of 3 miles per hour in still water. In a river with unknown current, it takes the boat twice as long to travel 60 miles upstream (against the current) than it takes for the 60 mile return trip (with the current). What is the speed of the current in the river?

The speed of the boat in still water is 3 miles per hour. When the boat travels upstream, the current is against the direction the boat is traveling and works to reduce the actual speed of the boat. When the boat travels downstream, then the actual speed of the boat is its speed in still water increased by the speed of the current. If we let \( c \) represent the speed of the current in the river, then the boat’s speed upstream (against the current) is \( 3 - c \), while the boat’s speed downstream (with the current) is \( 3 + c \). Let’s summarize what we know in a distance-speed-time table (see Table 1).
Section 7.8  Applications of Rational Functions

Table 1. A distance, speed, and time table.

<table>
<thead>
<tr>
<th></th>
<th>d (mi)</th>
<th>v (mi/h)</th>
<th>t (h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upstream</td>
<td>60</td>
<td>3 − c</td>
<td>?</td>
</tr>
<tr>
<td>Downstream</td>
<td>60</td>
<td>3 + c</td>
<td>?</td>
</tr>
</tbody>
</table>

Table 2. Calculating the time column entries.

<table>
<thead>
<tr>
<th></th>
<th>d (mi)</th>
<th>v (mi/h)</th>
<th>t (h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upstream</td>
<td>60</td>
<td>3 − c</td>
<td>( \frac{60}{3 - c} )</td>
</tr>
<tr>
<td>Downstream</td>
<td>60</td>
<td>3 + c</td>
<td>( \frac{60}{3 + c} )</td>
</tr>
</tbody>
</table>

Here is a useful piece of advice regarding distance, speed, and time tables.

**Distance, Speed, and Time Tables.** Because distance, speed, and time are related by the equation \( d = vt \), whenever you have two boxes in a row of the table completed, the third box in that row can be calculated by means of the formula \( d = vt \).

Note that each row of **Table 1** has two entries entered. The third entry in each row is time. Solve the equation \( d = vt \) for \( t \) to obtain

\[
t = \frac{d}{v}.
\]

The relation \( t = d/v \) can be used to compute the time entry in each row of **Table 1**.

For example, in the first row, \( d = 60 \) miles and \( v = 3 − c \) miles per hour. Therefore, the time of travel is

\[
t = \frac{d}{v} = \frac{60}{3 - c}.
\]

Note how we’ve filled in this entry in **Table 2**. In similar fashion, the time to travel downstream is calculated with

\[
t = \frac{d}{v} = \frac{60}{3 + c}.
\]

We’ve also added this entry to the time column in **Table 2**.

To set up an equation, we need to use the fact that the time to travel upstream is twice the time to travel downstream. This leads to the result

\[
\frac{60}{3 - c} = 2 \left( \frac{60}{3 + c} \right),
\]

or equivalently,
\[
\frac{60}{3-c} = \frac{120}{3+c}.
\]

Multiply both sides by the common denominator, in this case, \((3 - c)(3 + c)\).

\[
(3 - c)(3 + c) \left( \frac{60}{3-c} \right) = \left( \frac{120}{3+c} \right) (3 - c)(3 + c)
\]

\[
60(3 + c) = 120(3 - c)
\]

Expand each side of this equation.

\[
180 + 60c = 360 - 120c
\]

This equation is linear (no power of \(c\) other than 1). Hence, we want to isolate all terms containing \(c\) on one side of the equation. We add 120c to both sides of the equation, then subtract 180 from both sides of the equation.

\[
60c + 120c = 360 - 180
\]

From here, it is simple to solve for \(c\).

\[
180c = 180
\]

\[
c = 1.
\]

Hence, the speed of the current is 1 mile per hour.

It is important to check that the solution satisfies the constraints of the problem statement.

- If the speed of the boat in still water is 3 miles per hour and the speed of the current is 1 mile per hour, then the speed of the boat upstream (against the current) will be 2 miles per hour. It will take 30 hours to travel 60 miles at this rate.
- The speed of the boat as it goes downstream (with the current) will be 4 miles per hour. It will take 15 hours to travel 60 miles at this rate.

Note that the time to travel upstream (30 hours) is twice the time to travel downstream (15 hours), so our solution is correct.

Let’s look at another example.

**Example 6.** A speedboat can travel 32 miles per hour in still water. It travels 150 miles upstream against the current then returns to the starting location. The total time of the trip is 10 hours. What is the speed of the current?

Let \(c\) represent the speed of the current. Going upstream, the boat struggles against the current, so its net speed is \(32-c\) miles per hour. On the return trip, the boat benefits from the current, so its net speed on the return trip is \(32+c\) miles per hour. The trip each way is 150 miles. We’ve entered this data in Table 3.
Section 7.8 Applications of Rational Functions

<table>
<thead>
<tr>
<th></th>
<th>d (mi)</th>
<th>v (mi/h)</th>
<th>t (h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upstream</td>
<td>150</td>
<td>32 − c</td>
<td>?</td>
</tr>
<tr>
<td>Downstream</td>
<td>150</td>
<td>32 + c</td>
<td>?</td>
</tr>
</tbody>
</table>

Table 3. Entering the given data in a distance, speed, and time table.

Solving $d = vt$ for the time $t$,

$$t = \frac{d}{v}.$$  

In the first row of Table 3, we have $d = 150$ miles and $v = 32 − c$ miles per hour. Hence, the time it takes the boat to go upstream is given by

$$t = \frac{d}{v} = \frac{150}{32 − c}.$$  

Similarly, upon examining the data in the second row of Table 3, the time it takes the boat to return downstream to its starting location is

$$t = \frac{d}{v} = \frac{150}{32 + c}.$$  

These results are entered in Table 4.

<table>
<thead>
<tr>
<th></th>
<th>d (mi)</th>
<th>v (mi/h)</th>
<th>t (h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upstream</td>
<td>150</td>
<td>32 − c</td>
<td>150/(32 − c)</td>
</tr>
<tr>
<td>Downstream</td>
<td>150</td>
<td>32 + c</td>
<td>150/(32 + c)</td>
</tr>
</tbody>
</table>

Table 4. Calculating the time to go upstream and return.

Because the total time to go upstream and return is 10 hours, we can write

$$\frac{150}{32 − c} + \frac{150}{32 + c} = 10.$$  

Multiply both sides by the common denominator $(32 − c)(32 + c)$.

$$(32 − c)(32 + c) \left(\frac{150}{32 − c} + \frac{150}{32 + c}\right) = 10(32 − c)(32 + c)$$

$$150(32 + c) + 150(32 − c) = 10(1024 − c^2)$$  

We can make the numbers a bit smaller by noting that both sides of the last equation are divisible by 10.

$$15(32 + c) + 15(32 − c) = 1024 − c^2$$  

Expand, simplify, make one side zero, then factor.
\[480 + 15c + 480 - 15c = 1024 - c^2\]
\[960 = 1024 - c^2\]
\[0 = 64 - c^2\]
\[0 = (8 + c)(8 - c)\]

Using the zero product property, either
\[8 + c = 0 \quad \text{or} \quad 8 - c = 0,\]
providing two solutions for the current,
\[c = -8 \quad \text{or} \quad c = 8.\]

Discarding the negative answer (speed is a positive quantity in this case), the speed of the current is 8 miles per hour.

Does our answer make sense?

- Because the speed of the current is 8 miles per hour, the boat travels 150 miles upstream at a net speed of 24 miles per hour. This will take \(\frac{150}{24}\) or 6.25 hours.
- The boat travels downstream 150 miles at a net speed of 40 miles per hour. This will take \(\frac{150}{40}\) or 3.75 hours.

Note that the total time to go upstream and return is \(6.25 + 3.75\), or 10 hours.

Let’s look at another class of problems.

**Work Problems**

A nice application of rational functions involves the amount of work a person (or team of persons) can do in a certain amount of time. We can handle these applications involving work in a manner similar to the method we used to solve distance, speed, and time problems. Here is the guiding principle.

**Work, Rate, and Time.** The amount of work done is equal to the product of the rate at which work is being done and the amount of time required to do the work. That is,

\[\text{Work} = \text{Rate} \times \text{Time}.\]

For example, suppose that Emilia can mow lawns at a rate of 3 lawns per hour. After 6 hours,

\[\text{Work} = 3 \frac{\text{lawns}}{\text{hr}} \times 6 \text{ hr} = 18\text{ lawns}.\]

A second important concept is the fact that rates add. For example, if Emilia can mow lawns at a rate of 3 lawns per hour and Michele can mow the same lawns at a
rate of 2 lawns per hour, then together they can mow the lawns at a combined rate of 5 lawns per hour.

Let’s look at an example.

**Example 7.** Bill can finish a report in 2 hours. Maria can finish the same report in 4 hours. How long will it take them to finish the report if they work together?

A common misconception is that the times add in this case. That is, it takes Bill 2 hours to complete the report and it takes Maria 4 hours to complete the same report, so if Bill and Maria work together it will take 6 hours to complete the report. A little thought reveals that this result is nonsense. Clearly, if they work together, it will take them less time than it takes Bill to complete the report alone; that is, the combined time will surely be less than 2 hours.

However, as we saw above, the rates at which they are working will add. To take advantage of this fact, we set up what we know in a Work, Rate, and Time table (see Table 5).

- It takes Bill 2 hours to complete 1 report. This is reflected in the entries in the first row of Table 5.
- It takes Maria 4 hours to complete 1 report. This is reflected in the entries in the second row of Table 5.
- Let \( t \) represent the time it takes them to complete 1 report if they work together. This is reflected in the entries in the last row of Table 5.

<table>
<thead>
<tr>
<th></th>
<th>( w ) (reports)</th>
<th>( r ) (reports/h)</th>
<th>( t ) (h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bill</td>
<td>1</td>
<td>?</td>
<td>2</td>
</tr>
<tr>
<td>Maria</td>
<td>1</td>
<td>?</td>
<td>4</td>
</tr>
<tr>
<td>Together</td>
<td>1</td>
<td>?</td>
<td>( t )</td>
</tr>
</tbody>
</table>

**Table 5.** A work, rate, and time table.

We have advice similar to that given for distance, speed, and time tables.

**Work, Rate, and Time Tables.** Because work, rate, and time are related by the equation

\[
\text{Work} = \text{Rate} \times \text{Time},
\]

whenever you have two boxes in a row completed, the third box in that row can be calculated by means of the relation \( \text{Work} = \text{Rate} \times \text{Time} \).

In the case of Table 5, we can calculate the rate at which Bill is working by solving the equation \( \text{Work} = \text{Rate} \times \text{Time} \) for the Rate, then substitute Bill’s data from row one of Table 5.

\[
\text{Rate} = \frac{\text{Work}}{\text{Time}} = \frac{1 \text{ report}}{2 \text{ h}}.
\]
Thus, Bill is working at a rate of 1/2 report per hour. Note how we’ve entered this result in the first row of Table 6. Similarly, Maria is working at a rate of 1/4 report per hour, which we’ve also entered in Table 6.

We’ve let $t$ represent the time it takes them to write 1 report if they are working together (see Table 5), so the following calculation gives us the combined rate.

$$\text{Rate} = \frac{\text{Work}}{\text{Time}} = \frac{1 \text{ report}}{t \text{ h}}.$$ 

That is, together they work at a rate of $1/t$ reports per hour. This result is also recorded in Table 6.

<table>
<thead>
<tr>
<th></th>
<th>$w$ (reports)</th>
<th>$r$ (reports/h)</th>
<th>$t$ (h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bill</td>
<td>1</td>
<td>1/2</td>
<td>2</td>
</tr>
<tr>
<td>Maria</td>
<td>1</td>
<td>1/4</td>
<td>4</td>
</tr>
<tr>
<td>Together</td>
<td>1</td>
<td>$1/t$</td>
<td>$t$</td>
</tr>
</tbody>
</table>

Table 6. Calculating the Rate entries.

In our discussion above, we pointed out the fact that rates add. Thus, the equation we seek lies in the Rate column of Table 6. Bill is working at a rate of 1/2 report per hour and Maria is working at a rate of 1/4 report per hour. Therefore, their combined rate is $1/2 + 1/4$ reports per hour. However, the last row of Table 6 indicates that the combined rate is also $1/t$ reports per hour. Thus, 

$$\frac{1}{2} + \frac{1}{4} = \frac{1}{t}.$$ 

Multiply both sides of this equation by the common denominator 4$t$.

$$(4t) \left[ \frac{1}{2} + \frac{1}{4} \right] = \left[ \frac{1}{t} \right] (4t)$$

$$2t + t = 4,$$

This equation is linear (no power of $t$ other than 1) and is easily solved.

$$3t = 4$$

$$t = 4/3$$

Thus, it will take 4/3 of an hour to complete 1 report if Bill and Maria work together.

Again, it is very important that we check this result.

- We know that Bill does 1/2 reports per hour. In 4/3 of an hour, Bill will complete

$$\text{Work} = \frac{1}{2} \frac{\text{reports}}{\text{h}} \times \frac{4}{3} \text{h} = \frac{2}{3} \text{ reports}.$$ 

That is, Bill will complete 2/3 of a report.

- We know that Maria does 1/4 reports per hour. In 4/3 of an hour, Maria will complete
That is, Maria will complete 1/3 of a report.

Clearly, working together, Bill and Maria will complete 2/3 + 1/3 reports, that is, one full report.

Let’s look at another example.

**Example 8.** It takes Liya 7 more hours to paint a kitchen than it takes Hank to complete the same job. Together, they can complete the same job in 12 hours. How long does it take Hank to complete the job if he works alone?

Let $H$ represent the time it takes Hank to complete the job of painting the kitchen when he works alone. Because it takes Liya 7 more hours than it takes Hank, let $H + 7$ represent the time it takes Liya to paint the kitchen when she works alone. This leads to the entries in Table 7.

<table>
<thead>
<tr>
<th></th>
<th>$w$ (kitchens)</th>
<th>$r$ (kitchens/h)</th>
<th>$t$ (h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hank</td>
<td>1</td>
<td>?</td>
<td>$H$</td>
</tr>
<tr>
<td>Liya</td>
<td>1</td>
<td>?</td>
<td>$H + 7$</td>
</tr>
<tr>
<td>Together</td>
<td>1</td>
<td>?</td>
<td>12</td>
</tr>
</tbody>
</table>

**Table 7.** Entering the given data for Hank and Liya.

We can calculate the rate at which Hank is working alone by solving the equation $Work = Rate \times Time$ for the Rate, then substituting Hank’s data from row one of Table 7.

$$Rate = \frac{Work}{Time} = \frac{1\text{ kitchen}}{H\text{ hour}}$$

Thus, Hank is working at a rate of $1/H$ kitchens per hour. Similarly, Liya is working at a rate of $1/(H + 7)$ kitchens per hour. Because it takes them 12 hours to complete the task when working together, their combined rate is $1/12$ kitchens per hour. Each of these rates is entered in Table 8.

<table>
<thead>
<tr>
<th></th>
<th>$w$ (kitchens)</th>
<th>$r$ (kitchens/h)</th>
<th>$t$ (h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hank</td>
<td>1</td>
<td>$1/H$</td>
<td>$H$</td>
</tr>
<tr>
<td>Liya</td>
<td>1</td>
<td>$1/(H + 7)$</td>
<td>$H + 7$</td>
</tr>
<tr>
<td>Together</td>
<td>1</td>
<td>$1/12$</td>
<td>12</td>
</tr>
</tbody>
</table>

**Table 8.** Calculating the rates.

Because the rates add, we can write

$$\frac{1}{H} + \frac{1}{H + 7} = \frac{1}{12}.$$
Multiply both sides of this equation by the common denominator \(12H(H + 7)\).

\[
12H(H + 7) \left( \frac{1}{H} + \frac{1}{H + 7} \right) = \left( \frac{1}{12} \right) 12H(H + 7)
\]

\[
12(H + 7) + 12H = H(H + 7)
\]

Expand and simplify.

\[
12H + 84 + 12H = H^2 + 7H
\]

\[
24H + 84 = H^2 + 7H
\]

This last equation is nonlinear, so make one side zero by subtracting \(24H\) and 84 from both sides of the equation.

\[
0 = H^2 + 7H - 24H - 84
\]

\[
0 = H^2 - 17H - 84
\]

Note that \(ac = (1)(-84) = -84\). The integer pair \(\{4, -21\}\) has product \(-84\) and sums to \(-17\). Hence,

\[
0 = (H + 4)(H - 21).
\]

Using the zero product property, either

\[
H + 4 = 0 \quad \text{or} \quad H - 21 = 0,
\]

leading to the solutions

\[
H = -4 \quad \text{or} \quad H = 21.
\]

We eliminate the solution \(H = -4\) from consideration (it doesn’t take Hank negative time to paint the kitchen), so we conclude that it takes Hank 21 hours to paint the kitchen.

Does our solution make sense?

- It takes Hank 21 hours to complete the kitchen, so he is finishing \(1/21\) of the kitchen per hour.
- It takes Liya 7 hours longer than Hank to complete the kitchen, namely 28 hours, so she is finishing \(1/28\) of the kitchen per hour.

Together, they are working at a combined rate of

\[
\frac{1}{21} + \frac{1}{28} = \frac{4}{84} + \frac{3}{84} = \frac{7}{84} = \frac{1}{12},
\]

or \(1/12\) of a kitchen per hour. This agrees with the combined rate in Table 8.
7.8 Exercises

1. The sum of the reciprocals of two consecutive odd integers is $-\frac{16}{63}$. Find the two numbers.

2. The sum of the reciprocals of two consecutive odd integers is $\frac{28}{195}$. Find the two numbers.

3. The sum of the reciprocals of two consecutive integers is $-\frac{19}{90}$. Find the two numbers.

4. The sum of a number and its reciprocal is $\frac{41}{29}$. Find the number(s).

5. The sum of the reciprocals of two consecutive even integers is $\frac{5}{12}$. Find the two numbers.

6. The sum of the reciprocals of two consecutive integers is $\frac{19}{90}$. Find the two numbers.

7. The sum of a number and twice its reciprocal is $\frac{9}{2}$. Find the number(s).

8. The sum of a number and its reciprocal is $\frac{5}{2}$. Find the number(s).

9. The sum of the reciprocals of two consecutive even integers is $\frac{11}{90}$. Find the two numbers.

10. The sum of a number and twice its reciprocal is $\frac{17}{6}$. Find the number(s).

11. The sum of the reciprocals of two numbers is $15/8$, and the second number is 2 larger than the first. Find the two numbers.

12. The sum of the reciprocals of two numbers is $16/15$, and the second number is 1 larger than the first. Find the two numbers.

13. Moira can paddle her kayak at a speed of 2 mph in still water. She paddles 3 miles upstream against the current and then returns to the starting location. The total time of the trip is 9 hours. What is the speed (in mph) of the current? Round your answer to the nearest hundredth.

14. Boris is kayaking in a river with a 6 mph current. Suppose that he can kayak 4 miles upstream in the same amount of time as it takes him to kayak 9 miles downstream. Find the speed (mph) of Boris’s kayak in still water.

15. Jacob can paddle his kayak at a speed of 6 mph in still water. He paddles 5 miles upstream against the current and then returns to the starting location. The total time of the trip is 5 hours. What is the speed (in mph) of the current? Round your answer to the nearest hundredth.

16. Boris can paddle his kayak at a speed of 6 mph in still water. If he can paddle 5 miles upstream in the same amount of time as it takes his to paddle 9 miles downstream, what is the speed of the current?
17. Jacob is canoeing in a river with a 5 mph current. Suppose that he can canoe 4 miles upstream in the same amount of time as it takes him to canoe 8 miles downstream. Find the speed (mph) of Jacob’s canoe in still water.

18. The speed of a freight train is 16 mph slower than the speed of a passenger train. The passenger train travels 518 miles in the same time that the freight train travels 406 miles. Find the speed of the freight train.

19. The speed of a freight train is 20 mph slower than the speed of a passenger train. The passenger train travels 440 miles in the same time that the freight train travels 280 miles. Find the speed of the freight train.

20. Emily can paddle her canoe at a speed of 2 mph in still water. She paddles 5 miles upstream against the current and then returns to the starting location. The total time of the trip is 6 hours. What is the speed (in mph) of the current? Round your answer to the nearest hundredth.

21. Jacob is canoeing in a river with a 2 mph current. Suppose that he can canoe 2 miles upstream in the same amount of time as it takes him to canoe 5 miles downstream. Find the speed (mph) of Jacob’s canoe in still water.

22. Moira can paddle her kayak at a speed of 2 mph in still water. If she can paddle 4 miles upstream in the same amount of time as it takes her to paddle 8 miles downstream, what is the speed of the current?

23. Boris can paddle his kayak at a speed of 6 mph in still water. If he can paddle 5 miles upstream in the same amount of time as it takes his to paddle 10 miles downstream, what is the speed of the current?

24. The speed of a freight train is 19 mph slower than the speed of a passenger train. The passenger train travels 544 miles in the same time that the freight train travels 392 miles. Find the speed of the freight train.

25. It takes Jean 15 hours longer to complete an inventory report than it takes Sanjay. If they work together, it takes them 10 hours. How many hours would it take Sanjay if he worked alone?

26. Jean can paint a room in 5 hours. It takes Amelie 10 hours to paint the same room. How many hours will it take if they work together?

27. It takes Amelie 18 hours longer to complete an inventory report than it takes Jean. If they work together, it takes them 12 hours. How many hours would it take Jean if she worked alone?

28. Sanjay can paint a room in 5 hours. It takes Amelie 9 hours to paint the same room. How many hours will it take if they work together?

29. It takes Ricardo 12 hours longer to complete an inventory report than it takes Sanjay. If they work together, it takes them 8 hours. How many hours would it take Sanjay if he worked alone?
30. It takes Ricardo 8 hours longer to complete an inventory report than it takes Amelie. If they work together, it takes them 3 hours. How many hours would it take Amelie if she worked alone?

31. Jean can paint a room in 4 hours. It takes Sanjay 7 hours to paint the same room. How many hours will it take if they work together?

32. Amelie can paint a room in 5 hours. It takes Sanjay 9 hours to paint the same room. How many hours will it take if they work together?
7.8 Answers

1. $-9, -7$

3. $-10, -9$

5. $4, 6$

7. $\frac{1}{2}, 4$

9. $10, 12$

11. $\{\frac{2}{3}, \frac{8}{3}\}$ and $\{-\frac{8}{5}, \frac{2}{5}\}$

13. 1.63 mph

15. 4.90 mph

17. 15 mph

19. 35 mph

21. $\frac{14}{3}$ mph

23. 2 mph

25. 15 hours

27. 18 hours

29. 12 hours

31. $\frac{28}{11}$ hours
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