9 Radical Functions

In this chapter, we will study radical functions — in other words, functions that involve square, cubic, and other roots of algebraic expressions (for example, $\sqrt{x}$ or $\sqrt[3]{x + 2}$). There are a number of subtleties and tricks to these functions, and it is important to learn how to manipulate them.

Radical functions are closely related to power functions (for example, $x^2$ or $(2 - x)^5$). In fact, the graph of $\sqrt{x}$ is exactly what you would see if you reflected the graph of $x^2$ across the line $y = x$ and erased everything below the $x$-axis! It turns out that $\sqrt{x}$ is so closely related to $x^2$ that we say that those functions are inverses of each other; whatever one does, the other undoes.

Radical functions have many interesting applications, are studied extensively in many mathematics courses, and are used often in science and engineering. If you have ever wanted to calculate the shortest distance between two places, or predict how long a stairway is based upon the height it reaches, radical functions can help you with these calculations.

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9.1 The Square Root Function

In this section we turn our attention to the square root function, the function defined by the equation

\[ f(x) = \sqrt{x}. \]  

(1)

We begin the section by drawing the graph of the function, then we address the domain and range. After that, we’ll investigate a number of different transformations of the function.

**The Graph of the Square Root Function**

Let’s create a table of points that satisfy the equation of the function, then plot the points from the table on a Cartesian coordinate system on graph paper. We’ll continue creating and plotting points until we are convinced of the eventual shape of the graph.

We know we cannot take the square root of a negative number. Therefore, we don’t want to put any negative \(x\)-values in our table. To further simplify our computations, let’s use numbers whose square root is easily calculated. This brings to mind perfect squares such as 0, 1, 4, 9, and so on. We’ve placed these numbers as \(x\)-values in the table in Figure 1(b), then calculated the square root of each. In Figure 1(a), you see each of the points from the table plotted as a solid dot. If we continue to add points to the table, plot them, the graph will eventually fill in and take the shape of the solid curve shown in Figure 1(c).

\[
\begin{array}{c|c}
 x & f(x) = \sqrt{x} \\
 0 & 0 \\
 1 & 1 \\
 4 & 2 \\
 9 & 3 \\
\end{array}
\]

**Figure 1.** Creating the graph of \(f(x) = \sqrt{x}\).

The point plotting approach used to draw the graph of \(f(x) = \sqrt{x}\) in Figure 1 is a tested and familiar procedure. However, a more sophisticated approach involves the theory of inverses developed in the previous chapter.

In a sense, taking the square root is the “inverse” of squaring. Well, not quite, as the squaring function \(f(x) = x^2\) in Figure 2(a) fails the horizontal line test and is not one-to-one. However, if we limit the domain of the squaring function, then the graph of \(f(x) = x^2\) in Figure 2(b), where \(x \geq 0\), does pass the horizontal line test and is

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one-to-one. Therefore, the graph of $f(x) = x^2$, $x \geq 0$, has an inverse, and the graph of its inverse is found by reflecting the graph of $f(x) = x^2$, $x \geq 0$, across the line $y = x$ (see Figure 2(c)).

![Graphs](image)

(a) $f(x) = x^2$. (b) $f(x) = x^2$, $x \geq 0$. (c) Reflecting the graph in (b) across the line $y = x$ produces the graph of $f^{-1}(x) = \sqrt{x}$.

**Figure 2.** Sketching the inverse of $f(x) = x^2$, $x \geq 0$.

To find the equation of the inverse, recall that the procedure requires that we switch the roles of $x$ and $y$, then solve the resulting equation for $y$. Thus, first write $f(x) = x^2$, $x \geq 0$, in the form

$$y = x^2, \quad x \geq 0.$$  
Next, switch $x$ and $y$.

$$x = y^2, \quad y \geq 0 \quad (2)$$

When we solve this last equation for $y$, we get two solutions,

$$y = \pm \sqrt{x}. \quad (3)$$

However, in **equation (2)**, note that $y$ must be greater than or equal to zero. Hence, we must choose the nonnegative answer in **equation (3)**, so the inverse of $f(x) = x^2$, $x \geq 0$, has equation

$$f^{-1}(x) = \sqrt{x}.$$  
This is the equation of the reflection of the graph of $f(x) = x^2$, $x \geq 0$, that is pictured in Figure 2(c). Note the exact agreement with the graph of the square root function in Figure 1(c).

The sequence of graphs in Figure 2 also help us identify the domain and range of the square root function.
In Figure 2(a), the parabola opens outward indefinitely, both left and right. Consequently, the domain is $D_f = (-\infty, \infty)$, or all real numbers. Also, the graph has vertex at the origin and opens upward indefinitely, so the range is $R_f = [0, \infty)$. 

In Figure 2(b), we restricted the domain. Thus, the graph of $f(x) = x^2$, $x \geq 0$, now has domain $D_f = [0, \infty)$. The range is unchanged and is $R_f = [0, \infty)$. 

In Figure 2(c), we’ve reflected the graph of $f(x) = x^2$, $x \geq 0$ to obtain the graph of $f^{-1}(x) = \sqrt{x}$. Because we’ve interchanged the role of $x$ and $y$, the domain of the square root function must equal the range of $f(x) = x^2$, $x \geq 0$. That is, $D_{f^{-1}} = [0, \infty)$. Similarly, the range of the square root function must equal the domain of $f(x) = x^2$, $x \geq 0$. Hence, $R_{f^{-1}} = [0, \infty)$. 

Of course, we can also determine the domain and range of the square root function by projecting all points on the graph onto the $x$- and $y$-axes, as shown in Figures 3(a) and (b), respectively.

![Figure 3](image)

Figure 3. Project onto the axes to find the domain and range.

Some might object to the range, asking “How do we know that the graph of the square root function picture in Figure 3(b) rises indefinitely?” Again, the answer lies in the sequence of graphs in Figure 2. In Figure 2(c), note that the graph of $f(x) = x^2$, $x \geq 0$, opens indefinitely to the right as the graph rises to infinity. Hence, after reflecting this graph across the line $y = x$, the resulting graph must rise upward indefinitely as it moves to the right. Thus, the range of the square root function is $[0, \infty)$. 

Translations

If we shift the graph of $y = \sqrt{x}$ right and left, or up and down, the domain and/or range are affected.

**Example 4.** Sketch the graph of $f(x) = \sqrt{x-2}$. Use your graph to determine the domain and range.

We know that the basic equation $y = \sqrt{x}$ has the graph shown in Figure 1(c). If we replace $x$ with $x-2$, the basic equation $y = \sqrt{x}$ becomes $y = \sqrt{x-2}$. From our previous work with geometric transformations, we know that this will shift the graph two units to the right, as shown in Figures 4(a) and (b).
To find the domain, we project each point on the graph of $f$ onto the $x$-axis, as shown in Figure 4(a). Note that all points to the right of or including 2 are shaded on the $x$-axis. Consequently, the domain of $f$ is

$$\text{Domain} = [2, \infty) = \{x : x \geq 2\}.$$ 

As there has been no shift in the vertical direction, the range remains the same. To find the range, we project each point on the graph onto the $y$-axis, as shown in Figure 4(b). Note that all points at and above zero are shaded on the $y$-axis. Thus, the range of $f$ is

$$\text{Range} = [0, \infty) = \{y : y \geq 0\}.$$ 

We can find the domain of this function algebraically by examining its defining equation $f(x) = \sqrt{x - 2}$. We understand that we cannot take the square root of a negative number. Therefore, the expression under the radical must be nonnegative (positive or zero). That is,

$$x - 2 \geq 0.$$ 

Solving this inequality for $x$,

$$x \geq 2.$$ 

Thus, the domain of $f$ is Domain = $[2, \infty)$, which matches the graphical solution above.

Let’s look at another example.

**Example 5.** Sketch the graph of $f(x) = \sqrt{x + 4} + 2$. Use your graph to determine the domain and range of $f$.

Again, we know that the basic equation $y = \sqrt{x}$ has the graph shown in Figure 1(c). If we replace $x$ with $x + 4$, the basic equation $y = \sqrt{x}$ becomes $y = \sqrt{x + 4}$. From our
previous work with geometric transformations, we know that this will shift the graph
of \( y = \sqrt{x} \) four units to the left, as shown in Figure 5(a).

If we now add 2 to the equation \( y = \sqrt{x+4} \) to produce the equation \( y = \sqrt{x+4} + 2 \),
this will shift the graph of \( y = \sqrt{x+4} \) two units upward, as shown in Figure 5(b).

\[
\begin{align*}
\text{(a) To draw the graph of } y &= \sqrt{x+4}, \text{ shift the graph of } y = \sqrt{x} \\
\text{four units to the left.}
\end{align*}
\]

\[
\begin{align*}
\text{(b) To draw the graph of } y &= \sqrt{x+4} + 2, \text{ shift the graph of } y = \sqrt{x+4} \\
\text{two units upward.}
\end{align*}
\]

**Figure 5.** Translating the original equation \( y = \sqrt{x} \) to get the graph of \( y = \sqrt{x+4} + 2 \).

To identify the domain of \( f(x) = \sqrt{x+4} + 2 \), we project all points on the graph
of \( f \) onto the \( x \)-axis, as shown in Figure 6(a). Note that all points to the right of or
including \(-4\) are shaded on the \( x \)-axis. Thus, the domain of \( f(x) = \sqrt{x+4} + 2 \) is

\[
\text{Domain } = \left[-4, \infty\right) = \{x : x \geq -4\}.
\]

\[
\begin{align*}
\text{(a) Shading the domain of } f. \\
\text{(b) Shading the range of } f.
\end{align*}
\]

**Figure 6.** Project points of \( f \) onto the axes to determine the domain and range.

Similarly, to find the range of \( f \), project all points on the graph of \( f \) onto the \( y \)-axis,
as shown in Figure 6(b). Note that all points on the \( y \)-axis greater than or including
2 are shaded. Consequently, the range of \( f \) is
Range = \([2, \infty) = \{y : y \geq 2\}\).

We can also find the domain of \(f\) algebraically by examining the equation \(f(x) = \sqrt{x+4} + 2\). We cannot take the square root of a negative number, so the expression under the radical must be nonnegative (zero or positive). Consequently,

\[x + 4 \geq 0.\]

Solving this inequality for \(x\),

\[x \geq -4.\]

Thus, the domain of \(f\) is Domain = \([-4, \infty)\), which matches the graphical solution presented above.

**Reflections**

If we start with the basic equation \(y = \sqrt{x}\), then replace \(x\) with \(-x\), then the graph of the resulting equation \(y = \sqrt{-x}\) is captured by reflecting the graph of \(y = \sqrt{x}\) (see Figure 1(c)) horizontally across the \(y\)-axis. The graph of \(y = \sqrt{-x}\) is shown in Figure 7(a).

Similarly, the graph of \(y = -\sqrt{x}\) would be a vertical reflection of the graph of \(y = \sqrt{x}\) across the \(x\)-axis, as shown in Figure 7(b).

![Figure 7](image.png)

**Figure 7.** Reflecting the graph of \(y = \sqrt{x}\) across the \(x\)- and \(y\)-axes.

More often than not, you will be asked to perform a reflection and a translation.

**Example 6.** Sketch the graph of \(f(x) = \sqrt{4-x}\). Use the resulting graph to determine the domain and range of \(f\).
First, rewrite the equation \( f(x) = \sqrt{4-x} \) as follows:

\[
f(x) = \sqrt{-x-4}.
\]

**Reflections First.** It is usually more intuitive to perform reflections before translations.

With this thought in mind, we first sketch the graph of \( y = \sqrt{-x} \), which is a reflection of the graph of \( y = \sqrt{x} \) across the \( y \)-axis. This is shown in Figure 8(a).

Now, in \( y = \sqrt{-x} \), replace \( x \) with \( x - 4 \) to obtain \( y = \sqrt{-(x-4)} \). This shifts the graph of \( y = \sqrt{-x} \) four units to the right, as pictured in Figure 8(b).

![Figure 8](image)

**Figure 8.** A reflection followed by a translation.

To find the domain of the function \( f(x) = \sqrt{-(x-4)} \), or equivalently, \( f(x) = \sqrt{4-x} \), project each point on the graph of \( f \) onto the \( x \)-axis, as shown in Figure 9(a). Note that all real numbers less than or equal to 4 are shaded on the \( x \)-axis. Hence, the domain of \( f \) is

\[
\text{Domain} = (-\infty, 4] = \{x : x \leq 4\}.
\]

Similarly, to obtain the range of \( f \), project each point on the graph of \( f \) onto the \( y \)-axis, as shown in Figure 9(b). Note that all real numbers greater than or equal to zero are shaded on the \( y \)-axis. Hence, the range of \( f \) is

\[
\text{Range} = [0, \infty) = \{y : y \geq 0\}.
\]

We can also find the domain of the function \( f \) by examining the equation \( f(x) = \sqrt{4-x} \). We cannot take the square root of a negative number, so the expression under the radical must be nonnegative (zero or positive). Consequently,

\[
4 - x \geq 0.
\]
Solve this last inequality for $x$. First subtract 4 from both sides of the inequality, then multiply both sides of the resulting inequality by $-1$. Of course, multiplying by a negative number reverses the inequality symbol.

$$-x \geq -4$$
$$x \leq 4$$

Thus, the domain of $f$ is $\{x : x \leq 4\}$. In interval notation, $\text{Domain} = (-\infty, 4]$. This agree nicely with the graphical result found above.

More often than not, it will take a combination of your graphing calculator and a little algebraic manipulation to determine the domain of a square root function.

**Example 7.** Sketch the graph of $f(x) = \sqrt{5 - 2x}$. Use the graph and an algebraic technique to determine the domain of the function.

Load the function into Y1 in the Y= menu of your calculator, as shown in Figure 10(a). Select 6:ZStandard from the ZOOM menu to produce the graph shown in Figure 10(b).
Look carefully at the graph in Figure 10(b) and note that it’s difficult to tell if the graph comes all the way down to “touch” the x-axis near $x \approx 2.5$. However, our previous experience with the square root function makes us believe that this is just an artifact of insufficient resolution on the calculator that is preventing the graph from “touching” the x-axis at $x \approx 2.5$.

An algebraic approach will settle the issue. We can determine the domain of $f$ by examining the equation $f(x) = \sqrt{5 - 2x}$. We cannot take the square root of a negative number, so the expression under the radical must be nonnegative (zero or positive). Consequently,

$$5 - 2x \geq 0.$$ 

Solve this last inequality for $x$. First, subtract 5 from both sides of the inequality.

$$-2x \geq -5$$

Next, divide both sides of this last inequality by $-2$. Remember that we must reverse the inequality the moment we divide by a negative number.

$$\frac{-2x}{-2} \leq \frac{-5}{-2}$$

$$x \leq \frac{5}{2}$$

Thus, the domain of $f$ is $\{x : x \leq 5/2\}$. In interval notation, Domain $= (-\infty, 5/2]$.

Further introspection reveals that this argument also settles the issue of whether or not the graph “touches” the x-axis at $x = 5/2$. If you remain unconvinced, then substitute $x = 5/2$ in $f(x) = \sqrt{5 - 2x}$ to see

$$f(5/2) = \sqrt{5 - 2(5/2)} = \sqrt{0} = 0.$$ 

Thus, the graph of $f$ “touches” the x-axis at the point $(5/2, 0)$.
9.1 Exercises

In Exercises 1-10, complete each of the following tasks.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis.

ii. Complete the table of points for the given function. Plot each of the points on your coordinate system, then use them to help draw the graph of the given function.

iii. Use different colored pencils to project all points onto the $x$- and $y$-axes to determine the domain and range. Use interval notation to describe the domain of the given function.

1. $f(x) = -\sqrt{x}$

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<thead>
<tr>
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<th>4</th>
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2. $f(x) = \sqrt{-x}$

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3. $f(x) = \sqrt{x} + 2$

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4. $f(x) = \sqrt{5 - x}$

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5. $f(x) = \sqrt{x} + 2$

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6. $f(x) = \sqrt{x} - 1$

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7. $f(x) = \sqrt{x + 3} + 2$

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8. $f(x) = \sqrt{x - 1} + 3$

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9. $f(x) = \sqrt{3 - x}$

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10. $f(x) = -\sqrt{x + 3}$

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In **Exercises 11-20**, perform each of the following tasks.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*

ii. Use geometric transformations to draw the graph of the given function on your coordinate system without the use of a graphing calculator. *Note: You may check your solution with your calculator, but you should be able to produce the graph without the use of your calculator.*

iii. Use different colored pencils to project the points on the graph of the function onto the \( x \)- and \( y \)-axes. Use interval notation to describe the domain and range of the function.

11. \( f(x) = \sqrt{x} + 3 \)

12. \( f(x) = \sqrt{x} + 3 \)

13. \( f(x) = \sqrt{x} - 2 \)

14. \( f(x) = \sqrt{x} - 2 \)

15. \( f(x) = \sqrt{x} + 5 + 1 \)

16. \( f(x) = \sqrt{x} - 2 - 1 \)

17. \( f(x) = -\sqrt{x} + 4 \)

18. \( f(x) = -\sqrt{x} + 4 \)

19. \( f(x) = -\sqrt{x} + 3 \)

20. \( f(x) = -\sqrt{x} + 3 \)

21. To draw the graph of the function \( f(x) = \sqrt{3 - x} \), perform each of the following steps in sequence without the aid of a calculator.

i. Set up a coordinate system and sketch the graph of \( y = \sqrt{x} \). Label the graph with its equation.

ii. Set up a second coordinate system and sketch the graph of \( y = \sqrt{-x} \). Label the graph with its equation.

iii. Set up a third coordinate system and sketch the graph of \( y = \sqrt{-(x - 3)} \). Label the graph with its equation. This is the graph of \( f(x) = \sqrt{3 - x} \). Use interval notation to state the domain and range of this function.

22. To draw the graph of the function \( f(x) = \sqrt{-x - 3} \), perform each of the following steps in sequence.

i. Set up a coordinate system and sketch the graph of \( y = \sqrt{x} \). Label the graph with its equation.

ii. Set up a second coordinate system and sketch the graph of \( y = \sqrt{-x} \). Label the graph with its equation.

iii. Set up a third coordinate system and sketch the graph of \( y = \sqrt{-(x + 3)} \). Label the graph with its equation. This is the graph of \( f(x) = \sqrt{-x - 3} \). Use interval notation to state the domain and range of this function.

23. To draw the graph of the function \( f(x) = \sqrt{-x - 1} \), perform each of the following steps in sequence without the aid of a calculator.

i. Set up a coordinate system and sketch the graph of \( y = \sqrt{x} \). Label the graph with its equation.

ii. Set up a second coordinate system and sketch the graph of \( y = \sqrt{-x} \). Label the graph with its equation.

iii. Set up a third coordinate system and sketch the graph of \( y = \sqrt{-(x + 1)} \). Label the graph with its equation. This is the graph of \( f(x) = \sqrt{-x - 1} \). Use interval notation to state the domain and range of this function.
24. To draw the graph of the function $f(x) = \sqrt{1-x}$, perform each of the following steps in sequence.

i. Set up a coordinate system and sketch the graph of $y = \sqrt{x}$. Label the graph with its equation.
ii. Set up a second coordinate system and sketch the graph of $y = \sqrt{-x}$. Label the graph with its equation.
iii. Set up a third coordinate system and sketch the graph of $y = \sqrt{-x+1}$. Label the graph with its equation. This is the graph of $f(x) = \sqrt{1-x}$. Use interval notation to state the domain and range of this function.

In Exercises 29-40, find the domain of the given function algebraically.

29. $f(x) = \sqrt{2x+9}$
30. $f(x) = \sqrt{-3x+3}$
31. $f(x) = \sqrt{-8x-3}$
32. $f(x) = \sqrt{-3x+6}$
33. $f(x) = \sqrt{-6x-8}$
34. $f(x) = \sqrt{8x-6}$
35. $f(x) = \sqrt{-7x+2}$
36. $f(x) = \sqrt{8x-3}$
37. $f(x) = \sqrt{6x+3}$
38. $f(x) = \sqrt{2x-5}$
39. $f(x) = \sqrt{-7x-8}$
40. $f(x) = \sqrt{7x+8}$

In Exercises 25-28, perform each of the following tasks.

i. Draw the graph of the given function with your graphing calculator. Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label your graph with its equation. Use the graph to determine the domain of the function and describe the domain with interval notation.
ii. Use a purely algebraic approach to determine the domain of the given function. Use interval notation to describe your result. Does it agree with the graphical result from part (i)?

25. $f(x) = \sqrt{2x+7}$
26. $f(x) = \sqrt{7-2x}$
27. $f(x) = \sqrt{12-4x}$
28. $f(x) = \sqrt{12+2x}$
9.1 Answers

1. Domain = \([0, \infty)\), Range = \((-\infty, 0]\).

   \[
   \begin{array}{c|cccc}
   x & 0 & 1 & 4 & 9 \\
   \hline
   f(x) & 0 & -1 & -2 & -3 \\
   \end{array}
   \]

   \[
   \begin{array}{c|cccc}
   x & 0 & 1 & 4 & 9 \\
   \hline
   f(x) & 0 & -1 & -2 & -3 \\
   \end{array}
   \]

2. Domain = \([0, \infty)\), Range = \([-\infty, 0]\).

3. Domain = \([-2, \infty)\), Range = \([0, \infty)\).

   \[
   \begin{array}{c|cccc}
   x & -2 & -1 & 2 & 7 \\
   \hline
   f(x) & 0 & 1 & 2 & 3 \\
   \end{array}
   \]

4. Domain = \([-3, \infty)\), Range = \([2, \infty)\).

   \[
   \begin{array}{c|cccc}
   x & -3 & -2 & 1 & 6 \\
   \hline
   f(x) & 2 & 3 & 4 & 5 \\
   \end{array}
   \]

5. Domain = \([0, \infty)\), Range = \([2, \infty)\).

6. Domain = \([-3, \infty)\), Range = \([2, \infty)\).
9. Domain = \((-\infty, 3]\), Range = \([0, \infty)\).

<table>
<thead>
<tr>
<th>x</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

11. Domain = \([0, \infty)\), Range = \([3, \infty)\).

13. Domain = \([2, \infty)\), Range = \([0, \infty)\).

15. Domain = \([-5, \infty)\), Range = \([1, \infty)\).

17. Domain = \([-4, \infty)\), Range = \((-\infty, 0]\).

19. Domain = \([0, \infty)\), Range = \((-\infty, 3]\).
21. Domain = \((-\infty, 3]\), Range = \([0, \infty)\).  

23. Domain = \((-\infty, -1]\), Range = \([0, \infty)\).  

25. Domain = \([-7/2, \infty)\)
9.2 Multiplication Properties of Radicals

Recall that the equation \( x^2 = a \), where \( a \) is a positive real number, has two solutions, as indicated in Figure 1.

![Figure 1. The equation \( x^2 = a \), where \( a \) is a positive real number, has two solutions.](image_url)

Here are the key facts.

**Solutions of \( x^2 = a \).** If \( a \) is a positive real number, then:

1. The equation \( x^2 = a \) has two real solutions.
2. The notation \( \sqrt{a} \) denotes the **unique positive** real solution.
3. The notation \( -\sqrt{a} \) denotes the **unique negative** real solution.

Note the use of the word **unique**. When we say that \( \sqrt{a} \) is the unique positive real solution,\(^4\) we mean that it is the only one. There are no other positive real numbers that are solutions of \( x^2 = a \). A similar statement holds for the unique negative solution.

Thus, the equations \( x^2 = a \) and \( x^2 = b \) have unique positive solutions \( x = \sqrt{a} \) and \( x = \sqrt{b} \), respectively, provided that \( a \) and \( b \) are positive real numbers. Furthermore, because they are solutions, they can be substituted into the equations \( x^2 = a \) and \( x^2 = b \) to produce the results

\[
(\sqrt{a})^2 = a \quad \text{and} \quad (\sqrt{b})^2 = b,
\]

respectively. Again, these results are dependent upon the fact that \( a \) and \( b \) are positive real numbers.

Similarly, the equation

\(^3\) Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

\(^4\) Technically, the notation \( \sqrt{\cdot} \) calls for a **nonnegative** real square root, so as to include the possibility \( \sqrt{0} \).

Version: Fall 2007
\[ x^2 = ab \]

has unique positive solution \( x = \sqrt{ab} \), provided \( a \) and \( b \) are positive numbers. However, note that

\[
(\sqrt{a}\sqrt{b})^2 = (\sqrt{a})^2 (\sqrt{b})^2 = ab,
\]

making \( \sqrt{a}\sqrt{b} \) a **second** positive solution of \( x^2 = ab \). However, because \( \sqrt{ab} \) is the **unique** positive solution of \( x^2 = ab \), this forces

\[
\sqrt{ab} = \sqrt{a}\sqrt{b}.
\]

This discussion leads to the following property of radicals.

**Property 1.** Let \( a \) and \( b \) be positive real numbers. Then,

\[ \sqrt{ab} = \sqrt{a}\sqrt{b}. \quad (2) \]

This result can be used in two distinctly different ways.

- You can use the result to multiply two square roots, as in
  \[
  \sqrt{7}\sqrt{5} = \sqrt{35}.
  \]
- You can also use the result to factor, as in
  \[
  \sqrt{35} = \sqrt{5}\sqrt{7}.
  \]

It is interesting to check this result on the calculator, as shown in **Figure 2**.

![Figure 2](image)

**Figure 2.** Checking the result \( \sqrt{5}\sqrt{7} = \sqrt{35} \).

**Simple Radical Form**

In this section we introduce the concept of simple radical form, but let’s first start with a little story. Martha and David are studying together, working a homework problem from their textbook. Martha arrives at an answer of \( \sqrt{32} \), while David gets the result \( 2\sqrt{8} \). At first, David and Martha believe that their solutions are different numbers, but they’ve been mistaken before so they decide to compare decimal approximations of their results on their calculators. Martha’s result is shown in **Figure 3(a)**, while David’s is shown in **Figure 3(b)**.
Martha finds that $\sqrt{32} \approx 5.656854249$ and David finds that his solution $2\sqrt{8} \approx 5.656854249$. David and Martha conclude that their solutions match, but they want to know why the two very different looking radical expressions are identical.

The following calculation, using Property 1, shows why David’s result is identical to Martha’s.

$$\sqrt{32} = \sqrt{4\sqrt{8}} = 2\sqrt{8}$$

Indeed, there is even a third possibility, one that is much different from the results found by David and Martha. Consider the following calculation, which again uses Property 1.

$$\sqrt{32} = \sqrt{16\sqrt{2}} = 4\sqrt{2}$$

In Figure 4, note that the decimal approximation of $4\sqrt{2}$ is identical to the decimal approximations for $\sqrt{32}$ (Martha’s result in Figure 3(a)) and $2\sqrt{8}$ (David’s result in Figure 3(b)).

While all three of these radical expressions ($\sqrt{32}$, $2\sqrt{8}$, and $4\sqrt{2}$) are identical, it is somewhat frustrating to have so many different forms, particularly when we want to compare solutions. Therefore, we offer a set of guidelines for a special form of the answer which we will call simple radical form.

### The First Guideline for Simple Radical Form

When possible, factor out a perfect square.

Thus, $\sqrt{32}$ is not in simple radical form, as it is possible to factor out a perfect square, as in
\[ \sqrt{32} = \sqrt{16 \sqrt{2}} = 4\sqrt{2}. \]

Similarly, David’s result \((2\sqrt{8})\) is not in simple radical form, because he too can factor out a perfect square as follows.

\[ 2\sqrt{8} = 2(\sqrt{4 \sqrt{2}}) = 2(2\sqrt{2}) = (2 \cdot 2)\sqrt{2} = 4\sqrt{2}. \]

If both Martha and David follow the “first guideline for simple radical form,” their answers will look identical (both equal \(4\sqrt{2}\)). This is one of the primary advantages of simple radical form: the ability to compare solutions.

In the examples that follow (and in the exercises), it is helpful if you know the squares of the first 25 positive integers. We’ve listed them in the margin for you in Table 1 for future reference.

Let’s place a few more radical expressions in simple radical form.

\textbf{Example 3.} \textit{Place} \(\sqrt{50}\) \textit{in simple radical form.}

In Table 1, 25 is a square. Because \(50 = 25 \cdot 2\), we can use Property 1 to write

\[ \sqrt{50} = \sqrt{25\sqrt{2}} = 5\sqrt{2}. \]

\textbf{Example 4.} \textit{Place} \(\sqrt{98}\) \textit{in simple radical form.}

In Table 1, 49 is a square. Because \(98 = 49 \cdot 2\), we can again use Property 1 and write

\[ \sqrt{98} = \sqrt{49\sqrt{2}} = 7\sqrt{2}. \]

\textbf{Example 5.} \textit{Place} \(\sqrt{288}\) \textit{in simple radical form.}

Some students seem able to pluck the optimal “perfect square” out of thin air. If you consult Table 1, you’ll note that 144 is a square. Because \(288 = 144 \cdot 2\), we can write

\[ \sqrt{288} = \sqrt{144\sqrt{2}} = 12\sqrt{2}. \]

However, what if you miss that higher perfect square, think \(288 = 4 \cdot 72\), and write

\[ \sqrt{288} = \sqrt{4\sqrt{72}} = 2\sqrt{72}. \]

This approach is not incorrect, provided you realize that you’re not finished. You can still factor a perfect square out of 72. Because \(72 = 36 \cdot 2\), you can continue and write

\[ 2\sqrt{72} = 2(\sqrt{36\sqrt{2}}) = 2(6\sqrt{2}) = (2 \cdot 6)\sqrt{2} = 12\sqrt{2}. \]

Note that we arrived at the same simple radical form, namely \(12\sqrt{2}\). It just took us a little longer. As long as we realize that we must continue until we can no longer factor
out a perfect square, we’ll arrive at the same simple radical form as the student who
seems to magically pull the higher square out of thin air.

Indeed, here is another approach that is equally valid.

\[
\sqrt{288} = \sqrt{4\sqrt{72}} = 2(\sqrt{4\sqrt{18}}) = 2(2\sqrt{18}) = (2 \cdot 2)\sqrt{18} = 4\sqrt{18}
\]

We need to recognize that we are still not finished because we can extract another
perfect square as follows.

\[
4\sqrt{18} = 4(\sqrt{9}\sqrt{2}) = 4(3\sqrt{2}) = (4 \cdot 3)\sqrt{2} = 12\sqrt{2}
\]

Once again, same result. However, note that it behooves us to extract the largest
square possible, as it minimizes the number of steps required to attain simple radical
form.

**Checking Results with the Graphing Calculator.** Once you’ve placed a rad-
ical expression in simple radical form, you can use your graphing calculator to check
your result. In this example, we found that

\[
\sqrt{288} = 12\sqrt{2}.
\]

Enter the left- and right-hand sides of this result as shown in **Figure 5**. Note that each
side produces the same decimal approximation, verifying the result in **equation (6)**.

![Figure 5. Comparing \(\sqrt{288}\) with its
simple radical form \(12\sqrt{2}\).](image)

**Helpful Hints**

Recall that raising a power of a base to another power requires that we multiply expo-

ents.

**Raising a Power of a Base to another Power.**

\[
(a^m)^n = a^{mn}
\]

In particular, when you square a power of a base, you must multiply the exponent
by 2. For example,

\[
(2^5)^2 = 2^{10}.
\]
Conversely, because taking a square root is the “inverse” of squaring, when taking a square root we must divide the existing exponent by 2, as in
\[ \sqrt{2^{10}} = 2^5. \]
Note that squaring \( 2^5 \) gives \( 2^{10} \), so taking the square root of \( 2^{10} \) must return you to \( 2^5 \). When you square, you double the exponent. Therefore, when you take the square root, you must halve the exponent.

Similarly,
1. \((2^6)^2 = 2^{12}\) so \(\sqrt{2^{12}} = 2^6\).
2. \((2^7)^2 = 2^{14}\) so \(\sqrt{2^{14}} = 2^7\).
3. \((2^8)^2 = 2^{16}\) so \(\sqrt{2^{16}} = 2^8\).

This leads to the following result.

**Taking the Square Root of an Even Power.** When taking a square root of \(x^n\), when \(x\) is a positive real number and \(n\) is an even natural number, divide the exponent by two. In symbols,
\[ \sqrt{x^n} = x^{n/2}. \]
Note that this agrees with the definition of rational exponents presented in Chapter 8, as in
\[ \sqrt{x^n} = (x^n)^{1/2} = x^{n/2}. \]

On another note, recall that raising a product to a power requires that we raise each factor to that power.

**Raising a Product to a Power.**
\[ (ab)^n = a^n b^n. \]

In particular, if you square a product, you must square each factor. For example,
\[ (5^37^4)^2 = (5^3)^2(7^4)^2 = 5^67^8. \]
Note that we multiplied each existing exponent in this product by 2.

---

5 Well, not always. Consider \((-2)^2 = 4\), but \(\sqrt{4} = 2\) does not return to \(-2\). However, when you start with a positive number and square, then taking the positive square root is the inverse operation and returns you to the original positive number. Return to Chapter 8 (the section on inverse functions) if you want to reread a full discussion of this trickiness.
**Property 1** is similar, in that when we take the square root of a product, we take the square root of each factor. Because taking a square root is the inverse of squaring, we must *divide* each existing exponent by 2, as in

\[ \sqrt{5^6 \cdot 7^8} = \sqrt{5^6} \cdot \sqrt{7^8} = 5^3 \cdot 7^4. \]

Let’s look at some examples that employ this technique.

**Example 7.** Simplify \( \sqrt{2^4 \cdot 3^6 \cdot 5^{10}} \).

When taking the square root of a product of exponential factors, divide each exponent by 2.

\[ \sqrt{2^4 \cdot 3^6 \cdot 5^{10}} = 2^2 \cdot 3^3 \cdot 5^5 \]

If needed, you can expand the exponential factors and multiply to provide a single numerical answer.

\[ 2^2 \cdot 3^3 \cdot 5^5 = 4 \cdot 27 \cdot 3125 = 337500 \]

A calculator was used to obtain the final solution.

**Example 8.** Simplify \( \sqrt{2^5 \cdot 3^3} \).

In this example, the difficulty is the fact that the exponents are not divisible by 2. However, if possible, the “first guideline of simple radical form” requires that we factor out a perfect square. So, extract each factor raised to the highest possible power that is divisible by 2, as in

\[ \sqrt{2^5 \cdot 3^3} = \sqrt{2^4 \cdot 3^2} \cdot \sqrt{2 \cdot 3} \]

Now, divide each exponent by 2.

\[ \sqrt{2^4 \cdot 3^2} \cdot \sqrt{2 \cdot 3} = 2^2 \cdot 3^1 \cdot \sqrt{2 \cdot 3} \]

Finally, simplify by expanding each exponential factor and multiplying.

\[ 2^2 \cdot 3^1 \cdot \sqrt{2 \cdot 3} = 4 \cdot 3 \cdot \sqrt{6} = 12 \sqrt{6} \]

**Example 9.** Simplify \( \sqrt{3^7 \cdot 5^2 \cdot 7^5} \).

Extract each factor to the highest possible power that is divisible by 2.

\[ \sqrt{3^7 \cdot 5^2 \cdot 7^5} = \sqrt{3^6 \cdot 5^{2 \cdot 2} \cdot 7^4} \cdot \sqrt{3 \cdot 7} \]

Divide each exponent by 2.

\[ \sqrt{3^6 \cdot 5^{2 \cdot 2} \cdot 7^4} \cdot \sqrt{3 \cdot 7} = 3^3 \cdot 5^1 \cdot 7^2 \cdot \sqrt{3 \cdot 7} \]
Expand each exponential factor and multiply.

\[3^3 5^1 7^2 \sqrt{3 \cdot 7} = 27 \cdot 5 \cdot 49 \sqrt{3 \cdot 7} = 6,615 \sqrt{21}\]

**Example 10.** Place \(\sqrt{216}\) in simple radical form.

If we prime factor 216, we can attack this problem with the same technique used in the previous examples. Before we prime factor 216, here are a few divisibility tests that you might find useful.

**Divisibility Tests.**

- If a number ends in 0, 2, 4, 6, or 8, it is an **even** number and is divisible by 2.
- If the last two digits of a number form a number that is divisible by 4, then the entire number is divisible by 4.
- If a number ends in 0 or 5, it is divisible by 5.
- If the sum of the digits of a number is divisible by 3, then the entire number is divisible by 3.
- If the sum of the digits of a number is divisible by 9, then the entire number is divisible by 9.

For example, in order:

- The number 226 ends in a 6, so it is even and divisible by 2. Indeed, \(226 = 2 \cdot 113\).
- The last two digits of 224 are 24, which is divisible by 4, so the entire number is divisible by 4. Indeed, \(224 = 4 \cdot 56\).
- The last digit of 225 is a 5. Therefore 225 is divisible by 5. Indeed, \(225 = 5 \cdot 45\).
- The sum of the digits of 222 is \(2 + 2 + 2 = 6\), which is divisible by 3. Therefore, 222 is divisible by 3. Indeed, \(222 = 3 \cdot 74\).
- The sum of the digits of 684 is \(6 + 8 + 4 = 18\), which is divisible by 9. Therefore, 684 is divisible by 9. Indeed, \(684 = 9 \cdot 76\).

Now, let’s prime factor 216. Note that \(2 + 1 + 6 = 9\), so 216 is divisible by 9. Indeed, \(216 = 9 \cdot 24\). In **Figure 6**, we use a “factor tree” to continue factoring until all of the “leaves” are prime numbers.

![Figure 6](image)

**Figure 6.** Using a factor tree to prime factor 216.

Thus,
216 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3,

or in exponential form,

216 = 2^3 \cdot 3^3.

Thus,

\[ \sqrt{216} = \sqrt{2^3 \cdot 3^3} = \sqrt{2^2} \cdot \sqrt{3^2} \cdot \sqrt{2} = 2 \cdot 3 \sqrt{2} = 6 \sqrt{2}. \]

Prime factorization is an unbelievably useful tool!

Let’s look at another example.

**Example 11.** Place \( \sqrt{2592} \) in simple radical form.

If we find the prime factorization for 2592, we can attack this example using the same technique we used in the previous example. We note that the sum of the digits of 2592 is \( 2 + 5 + 9 + 2 = 18 \), which is divisible by 9. Therefore, 2592 is also divisible by 9.

2592 = 9 \cdot 288

The sum of the digits of 288 is \( 2 + 8 + 8 = 18 \), which is divisible by 9, so 288 is also divisible by 9.

2592 = 9 \cdot (9 \cdot 32)

Continue in this manner until the leaves of your “factor tree” are all primes. Then, you should get

2592 = 2^5 \cdot 3^4.

Thus,

\[ \sqrt{2592} = \sqrt{2^5 \cdot 3^4} = \sqrt{2^4 \cdot 2} \cdot \sqrt{3^4} \cdot \sqrt{3} = 2^2 \cdot 3 \cdot \sqrt{2} = 4 \cdot 9 \sqrt{2} = 36 \sqrt{2}. \]

Let’s use the graphing calculator to check this result. Enter each side of \( \sqrt{2592} = 36 \sqrt{2} \) separately and compare approximations, as shown in **Figure 7**.

**Figure 7.** Comparing \( \sqrt{2592} \) with its simple radical form \( 36 \sqrt{2} \).
An Important Property of Square Roots

One of the most common mistakes in algebra occurs when practitioners are asked to simplify the expression $\sqrt{x^2}$, where $x$ is any arbitrary real number. Let’s examine two of the most common errors.

- Some will claim that the following statement is true for any arbitrary real number $x$.

\[
\sqrt{x^2} = \pm x.
\]

This is easily seen to be incorrect. Simply substitute any real number for $x$ to check this claim. We will choose $x = 3$ and substitute it into each side of the proposed statement.

\[
\sqrt{3^2} = \pm 3
\]

If we simplify the left-hand side, we produce the following result.

\[
\sqrt{9} = \pm 3
\]

\[
3 = \pm 3
\]

It is not correct to state that 3 and $\pm 3$ are equal.

- A second error is to claim that

\[
\sqrt{x^2} = x
\]

for any arbitrary real number $x$. Although this is certainly true if you substitute nonnegative numbers for $x$, look what happens when you substitute $-3$ for $x$.

\[
\sqrt{(-3)^2} = -3
\]

If we simplify the left-hand side, we produce the following result.

\[
\sqrt{9} = -3
\]

\[
3 = -3
\]

Clearly, 3 and $-3$ are not equal.

In both cases, what has been forgotten is the fact that $\sqrt{}$ calls for a positive (nonnegative if you want to include the case $\sqrt{0}$) square root. In both of the errors above, namely $\sqrt{x^2} = \pm x$ and $\sqrt{x^2} = x$, the left-hand side is calling for a nonnegative response, but nothing has been done to insure that the right-hand side is also nonnegative. Does anything come to mind?

Sure, if we wrap the right-hand side in absolute values, as in

\[
\sqrt{x^2} = |x|,
\]
then both sides are calling for a nonnegative response. Indeed, note that
\[ \sqrt{(-3)^2} = |-3|, \quad \sqrt{0^2} = |0|, \quad \text{and} \quad \sqrt{3^2} = |3| \]
are all valid statements.

This discussion leads to the following result.

The Positive Square Root of the Square of \(x\). If \(x\) is any real number, then
\[ \sqrt{x^2} = |x|. \]

The next task is to use this new property to produce a extremely useful property of absolute value.

**A Multiplication Property of Absolute Value**

If we combine the law of exponents for squaring a product with our property for taking the square root of a product, we can write
\[ \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2} \sqrt{b^2}. \]
However, \(\sqrt{(ab)^2} = |ab|\), while \(\sqrt{a^2b^2} = |a||b|\). This discussion leads to the following result.

**Product Rule for Absolute Value.** If \(a\) and \(b\) are any real numbers,
\[ |ab| = |a||b|. \quad (12) \]
In words, the absolute value of a product is equal to the product of the absolute values.

We saw this property previously in the chapter on the absolute value function, where we provided a different approach to the proof of the property. It’s interesting that we can prove this property in a completely new way using the properties of square root. We’ll see we have need for the Product Rule for Absolute Value in the examples that follow.

For example, using the product rule, if \(x\) is any real number, we could write
\[ |3x| = |3||x| = 3|x| \]
However, there is no way we can remove the absolute value bars that surround \(x\) unless we know the sign of \(x\). If \(x \geq 0\), then \(|x| = x\) and the expression becomes
\[ 3|x| = 3x. \]
On the other hand, if \(x < 0\), then \(|x| = -x\) and the expression becomes
\[ 3|x| = 3(-x) = -3x. \]
Let’s look at another example. Using the product rule, if \( x \) is any real number, the expression \( | - 4x^3 | \) can be manipulated as follows.

\[
| - 4x^3 | = | - 4||x^2||x|
\]

However, \( | - 4 | = 4 \) and since \( x^2 \geq 0 \) for any value of \( x \), \( |x^2| = x^2 \). Thus,

\[
| - 4||x^2||x| = 4x^2|x|.
\]

Again, there is no way we can remove the absolute value bars around \( x \) unless we know the sign of \( x \). If \( x \geq 0 \), then \( |x| = x \) and

\[
4x^2|x| = 4x^2(x) = 4x^3.
\]

On the other hand, if \( x < 0 \), then \( |x| = -x \) and

\[
4x^2|x| = 4x^2(-x) = -4x^3.
\]

Let’s use these ideas to simplify some radical expressions that contain variables.

**Variable Expressions**

**Example 13.** Given that the \( x \) represents any real numbers, place the radical expression

\[
\sqrt{48x^6}
\]

in simple radical form.

Simple radical form demands that we factor out a perfect square, if possible. In this case, \( 48 = 16 \cdot 3 \) and we factor out the highest power of \( x \) that is divisible by 2.

\[
\sqrt{48x^6} = \sqrt{16x^6\sqrt{3}}
\]

We can now use **Property 1** to take the square root of each factor.

\[
\sqrt{16x^6\sqrt{3}} = \sqrt{16}\sqrt{x^6}\sqrt{3}
\]

Now, remember that the notation \( \sqrt{\text{\quad}} \) calls for a **nonnegative** square root, so we must insure that each response in the equation above is nonnegative. Thus,

\[
\sqrt{16}\sqrt{x^6}\sqrt{3} = 4|x^3|\sqrt{3}.
\]

Some comments are in order.

- The nonnegative square root of 16 is 4. That is, \( \sqrt{16} = 4 \).
- The nonnegative square root of \( x^6 \) is trickier. It is incorrect to say \( \sqrt{x^6} = x^3 \), because \( x^3 \) could be negative (if \( x \) is negative). To insure a nonnegative square root, in this case we need to wrap our answer in absolute value bars. That is, \( \sqrt{x^6} = |x^3| \).
We can use the Product Rule for Absolute Value to write \(|x^3| = |x^2|x|\). Because \(x^2\) is nonnegative, absolute value bars are redundant and not needed. That is, \(|x^2|x| = x^2|x|\). Thus, we can simplify our solution a bit further and write

\[ 4|x^3|\sqrt{3} = 4x^2|x|\sqrt{3}. \]

Thus,

\[ \sqrt{48x^6} = 4x^2|x|\sqrt{3}. \]  \hspace{1cm} (14)

**Alternate Solution.** There is a variety of ways that we can place a radical expression in simple radical form. Here is another approach. Starting at the step above, where we first factored out a perfect square,

\[ \sqrt{48x^6} = \sqrt{16x^6\sqrt{3}}, \]

we could write

\[ \sqrt{16x^6\sqrt{3}} = \sqrt{(4x^3)^2\sqrt{3}}. \]

Now, remember that the nonnegative square root of the square of an expression is the absolute value of that expression (we have to guarantee a nonnegative answer), so

\[ \sqrt{(4x^3)^2\sqrt{3}} = |4x^3|\sqrt{3}. \]

However, \(|4x^3| = |4||x^3|\) by our product rule and \(|4||x^3| = 4|x^3|\). Thus,

\[ |4x^3|\sqrt{3} = 4|x^3|\sqrt{3}. \]

Finally, \(|x^3| = |x^2|x| = x^2|x|\) because \(x^2 \geq 0\), so we can write

\[ 4|x^3|\sqrt{3} = 4x^2|x|\sqrt{3}. \]  \hspace{1cm} (15)

We cannot remove the absolute value bar that surrounds \(x\) unless we know the sign of \(x\).

Note that the simple radical form (15) in the alternate solution is identical to the simple radical form (14) found with the previous solution technique.

Let’s look at another example.

**Example 16.** Given that \(x < 0\), place \(\sqrt{24x^6}\) in simple radical form.

First, factor out a perfect square and write

\[ \sqrt{24x^6} = \sqrt{4x^6}\sqrt{6}. \]

Now, use Property 1 and take the square root of each factor.

\[ \sqrt{4x^6}\sqrt{6} = \sqrt{4x^6}\sqrt{6} \]

To insure a nonnegative response to \(\sqrt{x^6}\), wrap your response in absolute values.
\[ \sqrt{4\sqrt{x^6}\sqrt{6}} = 2|x^3|\sqrt{6} \]

However, as in the previous problem, \(|x^3| = |x^2||x| = x^2|x|\), since \(x^2 \geq 0\). Thus,

\[ 2|x^3|\sqrt{6} = 2x^2|x|\sqrt{6}. \]

In this example, we were given the extra fact that \(x < 0\), so \(|x| = -x\) and we can write

\[ 2x^2|x|\sqrt{6} = 2x^2(-x)\sqrt{6} = -2x^3\sqrt{6}. \]

It is instructive to test the validity of the answer

\[ \sqrt{24x^6} = -2x^3\sqrt{6}, \quad x < 0, \]

using a calculator. So, set \(x = -1\) with the command \(-1 \text{ STO} \uparrow X\). That is, enter \(-1\), then push the \(\text{STO} \uparrow\) button, followed by \(X\), then press the \(\text{ENTER}\) key. The result is shown in Figure 8(a). Next, enter \(\sqrt{-24*X^6}\) and press \(\text{ENTER}\) to capture the second result shown in Figure 8(a). Finally, enter \(-2*X^3\sqrt{6}\) and press \(\text{ENTER}\). Note that the two expressions \(\sqrt{24x^6}\) and \(-2x^3\sqrt{6}\) agree at \(x = -1\), as seen in Figure 8(a). We’ve also checked the validity of the result at \(x = -2\) in Figure 8(b). However, note that our result is not valid at \(x = 2\) in Figure 8(c). This occurs because \(\sqrt{24x^6} = -2x^3\sqrt{6}\) only if \(x\) is negative.

(a) Check with \(x = -1\). (b) Check with \(x = -2\). (c) Check with \(x = 2\).

Figure 8. Spot-checking the validity of \(\sqrt{24x^6} = -2x^3\sqrt{6}\).

It is somewhat counterintuitive that the result

\[ \sqrt{24x^6} = -2x^3\sqrt{6}, \quad x < 0, \]

contains a negative sign. After all, the expression \(\sqrt{24x^6}\) calls for a nonnegative result, but we have a negative sign. However, on closer inspection, if \(x < 0\), then \(x\) is a negative number and the right-hand side \(-2x^3\sqrt{6}\) is a positive number (\(-2\) is negative, \(x^3\) is negative because \(x\) is negative, and the product of two negatives is a positive).

Let’s look at another example.

**Example 17.** If \(x < 3\), simplify \(\sqrt{x^2 - 6x + 9}\).

The expression under the radical is a perfect square trinomial and factors.
\[ \sqrt{x^2 - 6x + 9} = \sqrt{(x - 3)^2} \]

However, the nonnegative square root of the square of an expression is the absolute value of that expression, so

\[ \sqrt{(x - 3)^2} = |x - 3|. \]

Finally, because we are told that \( x < 3 \), this makes \( x - 3 \) a negative number, so

\[ |x - 3| = -(x - 3). \quad (18) \]

Again, the result \( \sqrt{x^2 - 6x + 9} = -(x - 3) \), provided \( x < 3 \), is somewhat counterintuitive as we are expecting a positive result. However, if \( x < 3 \), the result \( -(x - 3) \) is positive. You can test this by substituting several values of \( x \) that are less than 3 into the expression \( -(x - 3) \) and noting that the result is positive. For example, if \( x = 2 \), then \( x \) is less than 3 and

\[ -(x - 3) = -(2 - 3) = -(-1) = 1, \]

which, of course, is a positive result.

It is even more informative to note that our result is equivalent to

\[ \sqrt{x^2 - 6x + 9} = -x + 3, \quad x < 3. \]

This is easily seen by distributing the minus sign in the result \( (18) \).

We’ve drawn the graph of \( y = \sqrt{x^2 - 6x + 9} \) on our calculator in Figure 9(a). In Figure 9(b), we’ve drawn the graph of \( y = -x + 3 \). Note that the graphs agree when \( x < 3 \). Indeed, when you consider the left-hand branch of the “V” in Figure 9(a), you can see that the slope of this branch is \(-1\) and the \( y \)-intercept is 3. The equation of this branch is \( y = -x + 3 \), so it agrees with the graph of \( y = -x + 3 \) in Figure 9(b) when \( x \) is less than 3.

![Graphs of y = \sqrt{x^2 - 6x + 9} and y = -x + 3](image)

(a) The graph of \( y = \sqrt{x^2 - 6x + 9} \).
(b) The graph of \( y = -x + 3 \).

**Figure 9.** Verifying graphically that \( \sqrt{x^2 - 6x + 9} = -x + 3 \) when \( x < 3 \).
9.2 Exercises

1. Use a calculator to first approximate $\sqrt{5}\sqrt{2}$. On the same screen, approximate $\sqrt{10}$. Report the results on your homework paper.

2. Use a calculator to first approximate $\sqrt{7}\sqrt{10}$. On the same screen, approximate $\sqrt{70}$. Report the results on your homework paper.

3. Use a calculator to first approximate $\sqrt{3}\sqrt{11}$. On the same screen, approximate $\sqrt{33}$. Report the results on your homework paper.

4. Use a calculator to first approximate $\sqrt{5}\sqrt{13}$. On the same screen, approximate $\sqrt{65}$. Report the results on your homework paper.

In Exercises 5-20, place each of the radical expressions in simple radical form. As in Example 3 in the narrative, check your result with your calculator.

5. $\sqrt{18}$
6. $\sqrt{80}$
7. $\sqrt{112}$
8. $\sqrt{72}$
9. $\sqrt{108}$
10. $\sqrt{54}$
11. $\sqrt{50}$
12. $\sqrt{48}$
13. $\sqrt{245}$
14. $\sqrt{150}$
15. $\sqrt{98}$
16. $\sqrt{252}$
17. $\sqrt{45}$
18. $\sqrt{294}$
19. $\sqrt{24}$
20. $\sqrt{32}$

In Exercises 21-26, use prime factorization (as in Examples 10 and 11 in the narrative) to assist you in placing the given radical expression in simple radical form. Check your result with your calculator.

21. $\sqrt{2016}$
22. $\sqrt{2700}$
23. $\sqrt{14175}$
24. $\sqrt{44000}$
25. $\sqrt{20250}$
26. $\sqrt{3564}$

In Exercises 27-46, place each of the given radical expressions in simple radical form. Make no assumptions about the sign of the variables. Variables can either represent positive or negative numbers.

27. $\sqrt{(6x - 11)^4}$

6 Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/
28. \( \sqrt{16h^8} \)

29. \( \sqrt{25f^2} \)

30. \( \sqrt{25j^8} \)

31. \( \sqrt{16m^2} \)

32. \( \sqrt{25a^2} \)

33. \( \sqrt{(7x + 5)^{12}} \)

34. \( \sqrt{9w^{10}} \)

35. \( \sqrt{25x^2 - 50x + 25} \)

36. \( \sqrt{49x^2 - 42x + 9} \)

37. \( \sqrt{25x^2 + 90x + 81} \)

38. \( \sqrt{25j^{14}} \)

39. \( \sqrt{(3x + 6)^{12}} \)

40. \( \sqrt{(9x - 8)^{12}} \)

41. \( \sqrt{36x^2 + 36x + 9} \)

42. \( \sqrt{4e^2} \)

43. \( \sqrt{4p^{10}} \)

44. \( \sqrt{25x^{12}} \)

45. \( \sqrt{25q^6} \)

46. \( \sqrt{16h^{12}} \)

47. Given that \( x < 0 \), place the radical expression \( \sqrt{32x^6} \) in simple radical form. Check your solution on your calculator for \( x = -2 \).

48. Given that \( x < 0 \), place the radical expression \( \sqrt{54x^9} \) in simple radical form. Check your solution on your calculator for \( x = -2 \).

49. Given that \( x < 0 \), place the radical expression \( \sqrt{27x^{12}} \) in simple radical form. Check your solution on your calculator for \( x = -2 \).

50. Given that \( x < 0 \), place the radical expression \( \sqrt{44x^{10}} \) in simple radical form. Check your solution on your calculator for \( x = -2 \).

In Exercises 51-54, follow the lead of Example 17 in the narrative to simplify the given radical expression and check your result with your graphing calculator.

51. Given that \( x < 4 \), place the radical expression \( \sqrt{x^2 - 8x + 16} \) in simple radical form. Use a graphing calculator to show that the graphs of the original expression and your simple radical form agree for all values of \( x \) such that \( x < 4 \).

52. Given that \( x \geq -2 \), place the radical expression \( \sqrt{x^2 + 4x + 4} \) in simple radical form. Use a graphing calculator to show that the graphs of the original expression and your simple radical form agree for all values of \( x \) such that \( x \geq -2 \).

53. Given that \( x \geq 5 \), place the radical expression \( \sqrt{x^2 - 10x + 25} \) in simple radical form. Use a graphing calculator to show that the graphs of the original expression and your simple radical form agree for all values of \( x \) such that \( x \geq 5 \).

54. Given that \( x < -1 \), place the radical expression \( \sqrt{x^2 + 2x + 1} \) in simple radical form. Use a graphing calculator to show that the graphs of the original expression and your simple radical form agree for all values of \( x \) such that \( x < -1 \).
In Exercises 55-72, place each radical expression in simple radical form. Assume that all variables represent positive numbers.

55. $\sqrt[6]{9d^{13}}$

56. $\sqrt{4k^2}$

57. $\sqrt{25x^2 + 40x + 16}$

58. $\sqrt{9x^2 - 30x + 25}$

59. $\sqrt{4j^{11}}$

60. $\sqrt{16j^6}$

61. $\sqrt{25m^2}$

62. $\sqrt{9e^9}$

63. $\sqrt{4e^5}$

64. $\sqrt{25z^2}$

65. $\sqrt{25h^{10}}$

66. $\sqrt{25b^2}$

67. $\sqrt{9s^7}$

68. $\sqrt{9e^7}$

69. $\sqrt{4p^8}$

70. $\sqrt{9d^{15}}$

71. $\sqrt{9q^{10}}$

72. $\sqrt{4w^7}$

In Exercises 73-80, place each given radical expression in simple radical form. Assume that all variables represent positive numbers.

73. $\sqrt{2f^5} \sqrt{8f^3}$

74. $\sqrt{3s^3} \sqrt{243s^3}$

75. $\sqrt{2k^7} \sqrt{32k^3}$

76. $\sqrt{2n^9} \sqrt{8n^3}$

77. $\sqrt{2e^9} \sqrt{8e^3}$

78. $\sqrt{5n^9} \sqrt{125n^3}$

79. $\sqrt{3z^5} \sqrt{27z^3}$

80. $\sqrt{3l^7} \sqrt{27l^3}$
9.2 Answers

1. $\sqrt{5} \times \sqrt{10} = 3.16227766$

3. $\sqrt[3]{3} \times \sqrt[11]{33} = 5.744562647$

5. $3\sqrt{2}$

7. $4\sqrt{7}$

9. $6\sqrt{3}$

11. $5\sqrt{2}$

13. $7\sqrt{5}$

15. $7\sqrt{2}$

17. $3\sqrt{5}$

19. $2\sqrt{6}$

21. $12\sqrt{14}$

23. $45\sqrt{7}$

25. $45\sqrt{10}$

27. $(6x - 11)^2$

29. $5|f|$

31. $4|m|$

33. $(7x + 5)^6$

35. $|5x - 5|$

37. $|5x + 9|$

39. $(3x + 6)^6$

41. $|6x + 3|$

43. $2p^4|p|$

45. $5q^2|q|$

47. $-4x^3\sqrt{2}$

49. $3x^6\sqrt{3}$
51. \(-x + 4\). The graphs of \(y = -x + 4\) and \(y = \sqrt{x^2 - 8x + 16}\) follow. Note that they agree for \(x < 4\).

53. \(x - 5\). The graphs of \(y = x - 5\) and \(y = \sqrt{x^2 - 10x + 25}\) follow. Note that they agree for \(x \geq 5\).

55. \(3d^6\sqrt{d}\)

57. \(5x + 4\)

59. \(2j^5\sqrt{j}\)

61. \(5m\)

63. \(2c^2\sqrt{c}\)

65. \(5h^5\)

67. \(3s^3\sqrt{s}\)

69. \(2p^4\)

71. \(3q^5\)

73. \(4f^4\)

75. \(8k^5\)

77. \(4e^6\)

79. \(9z^4\)
9.3 Division Properties of Radicals

Each of the equations \( x^2 = a \) and \( x^2 = b \) has a unique positive solution, \( x = \sqrt{a} \) and \( x = \sqrt{b} \), respectively, provided \( a \) and \( b \) are positive real numbers. Further, because they are solutions, they can be substituted into the equations \( x^2 = a \) and \( x^2 = b \) to produce the results

\[
(\sqrt{a})^2 = a \quad \text{and} \quad (\sqrt{b})^2 = b,
\]

respectively. These results are dependent upon the fact that \( a \) and \( b \) are positive real numbers.

Similarly, the equation

\[ x^2 = \frac{a}{b} \]

has the unique positive solution

\[ x = \sqrt{\frac{a}{b}} \]

provided \( a \) and \( b \) are positive real numbers. However, note that

\[
\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{(\sqrt{a})^2}{(\sqrt{b})^2} = \frac{a}{b},
\]

making \( \sqrt{a}/\sqrt{b} \) a second positive solution of \( x^2 = a/b \). However, because \( \sqrt{a/b} \) is the unique positive solution of \( x^2 = a/b \), this forces

\[ \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}. \]

This discussion leads us to the following property of radicals.

**Property 1.** Let \( a \) and \( b \) be positive real numbers. Then,

\[ \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}. \]

This result can be used in two distinctly different ways.

- You can use the result to divide two square roots, as in

\[
\frac{\sqrt{13}}{\sqrt{7}} = \sqrt{\frac{13}{7}}.
\]
• You can also use the result to take the square root of a fraction. Simply take the square root of both numerator and denominator, as in

\[
\sqrt{\frac{13}{7}} = \frac{\sqrt{13}}{\sqrt{7}}.
\]

It is interesting to check these results on a calculator, as shown in Figure 1.

![Figure 1. Checking that \(\sqrt{\frac{13}{7}} = \frac{\sqrt{13}}{\sqrt{7}}\).](image)

**Simple Radical Form Continued**

David and Martha are again working on a homework problem. Martha obtains the solution \(\sqrt{\frac{1}{12}}\), but David’s solution \(1/(2\sqrt{3})\) is seemingly different. Having learned their lesson in an earlier assignment, they use their calculators to find decimal approximations of their solutions. Martha’s approximation is shown in Figure 2(a) and David’s approximation is shown in Figure 2(b).

![Figure 2. Comparing Martha’s \(\sqrt{\frac{1}{12}}\) with David’s \(1/(2\sqrt{3})\).](image)

Martha finds that \(\sqrt{\frac{1}{12}} \approx 0.2886751346\) and David finds that \(1/(2\sqrt{3}) \approx 0.2886751346\). They conclude that their answers match, but they want to know why such different looking answers are identical.

The following calculation shows why Martha’s result is identical to David’s. First, use the division property of radicals (*Property 1*) to take the square root of both numerator and denominator.

\[
\sqrt{\frac{1}{12}} = \frac{\sqrt{1}}{\sqrt{12}} = \frac{1}{\sqrt{12}}
\]
Next, use the “first guideline for simple radical form” and factor a perfect square from
the denominator.
\[
\frac{1}{\sqrt{12}} = \frac{1}{\sqrt{4\sqrt{3}}} = \frac{1}{2\sqrt{3}}
\]
This clearly demonstrates that David and Martha’s solutions are identical.

Indeed, there are other possible forms for the solution of David and Martha’s home-
work exercise. Start with Martha’s solution, then multiply both numerator and de-
nominator of the fraction under the radical by 3.
\[
\sqrt{\frac{1}{12}} = \sqrt{\frac{1}{12} \cdot \frac{3}{3}} = \sqrt{\frac{3}{36}}
\]
Now, use the division property of radicals (Property 1), taking the square root of
both numerator and denominator.
\[
\sqrt{\frac{3}{36}} = \frac{\sqrt{3}}{\sqrt{36}} = \frac{\sqrt{3}}{6}
\]
Note that the approximation of \(\sqrt{3}/6\) in Figure 3 is identical to Martha’s and David’s
approximations in Figures 2(a) and (b).

While all three of the solution forms (\(\sqrt{1/12}\), \(1/(2\sqrt{3})\), and \(\sqrt{3}/6\)) are identical,
it is very frustrating to have so many forms, particularly when we want to compare
solutions. So, we are led to establish two more guidelines for simple radical form.

The Second Guideline for Simple Radical Form. Don’t leave fractions under
a radical.

Thus, Martha’s \(\sqrt{1/12}\) is not in simple radical form, because it contains a fraction
under the radical.

The Third Guideline for Simple Radical Form. Don’t leave radicals in the
denominator of a fraction.
Thus, David’s $1/(2\sqrt{3})$ is not in simple radical form, because the denominator of his fraction contains a radical.

Only the equivalent form $\sqrt{3}/6$ obeys all three rules of simple radical form.

1. It is not possible to factor a perfect square from any radical in the expression $\sqrt{3}/6$.
2. There are no fractions under a radical in the expression $\sqrt{3}/6$.
3. The denominator in the expression $\sqrt{3}/6$ contains no radicals.

In this text and in this course, we will always follow the three guidelines for simple radical form.

**Simple Radical Form.** When your answer is a radical expression:

1. If possible, factor out a perfect square.
2. Don’t leave fractions under a radical.
3. Don’t leave radicals in the denominator of a fraction.

In the examples that follow (and in the exercises), it is helpful if you know the squares of the first 25 positive integers. We’ve listed them in the margin for you in **Table 1** for future reference.

Let’s place a few radical expressions in simple radical form. We’ll start with some radical expressions that contain fractions under a radical.

**Example 2.** Place the expression $\sqrt{1/8}$ in simple radical form.

The expression $\sqrt{1/8}$ contains a fraction under a radical. We could take the square root of both numerator and denominator, but that would produce $\sqrt{1}/\sqrt{8}$, which puts a radical in the denominator.

The better strategy is to change the form of $1/8$ so that we have a perfect square in the denominator before taking the square root of the numerator and denominator. We note that if we multiply 8 by 2, the result is 16, a perfect square. This is hopeful, so we begin the simplification by multiplying both numerator and denominator of $1/8$ by 2.

$$\sqrt{\frac{1}{8}} = \sqrt{\frac{1}{8} \cdot \frac{2}{2}} = \sqrt{\frac{2}{16}}$$

We now take the square root of both numerator and denominator. Because the denominator is now a perfect square, the result will not have a radical in the denominator.

$$\sqrt{\frac{2}{16}} = \frac{\sqrt{2}}{\sqrt{16}} = \frac{\sqrt{2}}{4}$$

---

8 In some courses, such as trigonometry and calculus, your instructor may relax these guidelines a bit. In some cases, it is easier to work with $1/\sqrt{2}$, for example, than it is to work with $\sqrt{2}/2$, even though they are equivalent.
This last result, $\sqrt{2}/4$ is in simple radical form. It is not possible to factor a perfect square from any radical, there are no fractions under any radical, and the denominator is free of radicals.

You can easily check your solution by using your calculator to compare the original expression with your simple radical form. In Figure 4(a), we’ve approximated the original expression, $\sqrt{1}/8$. In Figure 4(b), we’ve approximated our simple radical form, $\sqrt{2}/4$. Note that they yield identical decimal approximations.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Comparing $\sqrt{1}/8$ and $\sqrt{2}/4$.}
\end{figure}

Let’s look at another example.

\begin{example}
Place $\sqrt{3}/20$ in simple radical form.
\end{example}

Following the lead from Example 2, we note that $5 \cdot 20 = 100$, a perfect square. So, we multiply both numerator and denominator by 5, then take the square root of both numerator and denominator once we have a perfect square in the denominator.

$$
\sqrt{\frac{3}{20}} = \sqrt{\frac{3 \cdot 5}{20 \cdot 5}} = \sqrt{\frac{15}{100}} = \frac{\sqrt{15}}{\sqrt{100}} = \frac{\sqrt{15}}{10}
$$

Note that the decimal approximation of the simple radical form $\sqrt{15}/10$ in Figure 5(b) matches the decimal approximation of the original expression $\sqrt{3}/20$ in Figure 5(a).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Comparing the original $\sqrt{3}/20$ with the simple radical form $\sqrt{15}/10$.}
\end{figure}
We will now show how to deal with an expression having a radical in its denominator, but first we pause to explain a new piece of terminology.

**Rationalizing the Denominator.** The process of eliminating radicals from the denominator is called rationalizing the denominator because it results in a fraction where the denominator is free of radicals and is a rational number.

**Example 4.** Place the expression \( \frac{5}{\sqrt{18}} \) in simple radical form.

In the previous examples, making the denominator a perfect square seemed a good tactic. We apply the same tactic in this example, noting that \( 2 \cdot 18 = 36 \) is a perfect square. However, the strategy is slightly different, as we begin the solution by multiplying both numerator and denominator by \( \sqrt{2} \).

\[
\frac{5}{\sqrt{18}} = \frac{5}{\sqrt{18}} \cdot \sqrt{2} \cdot \sqrt{2}
\]

We now multiply numerators and denominators. In the denominator, the multiplication property of radicals is used, \( \sqrt{18} \sqrt{2} = \sqrt{36} \).

\[
\frac{5}{\sqrt{18}} \cdot \sqrt{2} = \frac{5\sqrt{2}}{\sqrt{36}}
\]

The strategy should now be clear. Because the denominator is a perfect square, \( \sqrt{36} = 6 \), clearing all radicals from the denominator of our result.

\[
\frac{5\sqrt{2}}{\sqrt{36}} = \frac{5\sqrt{2}}{6}
\]

The last result is in simple radical form. It is not possible to extract a perfect square root from any radical, there are no fractions under any radical, and the denominator is free of radicals.

In Figure 6, we compare the approximation for our original expression \( \frac{5}{\sqrt{18}} \) with our simple radical form \( \frac{5\sqrt{2}}{6} \).

![Figure 6](image)

Figure 6. Comparing \( \frac{5}{\sqrt{18}} \) with \( \frac{5\sqrt{2}}{6} \).

Let’s look at another example.
Example 5. Place the expression $18/\sqrt{27}$ in simple radical form.

Note that $3 \cdot 27 = 81$ is a perfect square. We begin by multiplying both numerator and denominator of our expression by $\sqrt{3}$.

\[
\frac{18}{\sqrt{27}} = \frac{18}{\sqrt{27}} \cdot \frac{\sqrt{3}}{\sqrt{3}}
\]

Multiply numerators and denominators. In the denominator, $\sqrt{27} \cdot \sqrt{3} = \sqrt{81}$.

\[
\frac{18 \cdot \sqrt{3}}{\sqrt{27} \cdot \sqrt{3}} = \frac{18\sqrt{3}}{\sqrt{81}}
\]

Of course, $\sqrt{81} = 9$, so

\[
\frac{18\sqrt{3}}{9} = \frac{18\sqrt{3}}{9}
\]

We can now reduce to lowest terms, dividing numerator and denominator by 9.

\[
\frac{18\sqrt{3}}{9} = 2\sqrt{3}
\]

In Figure 7, we compare approximations of the original expression $18/\sqrt{27}$ and its simple radical form $2\sqrt{3}$.

![Figure 7](image)

Figure 7. Comparing $18/\sqrt{27}$ with its simple radical form $2\sqrt{3}$.

Helpful Hints

In the previous section, we learned that if you square a product of exponential expressions, you multiply each of the exponents by 2.

\[(2^3 \cdot 3^5)^2 = 2^6 \cdot 3^{10}\]

Because taking the square root is the “inverse” of squaring,\(^9\) we divide each of the exponents by 2.

\[^9\text{As we have pointed out in previous sections, taking the positive square root is the inverse of squaring, only if we restrict the domain of the squaring function to nonnegative real numbers, which we do here.}\]
We also learned that prime factorization is an extremely powerful tool that is quite useful when placing radical expressions in simple radical form. We’ll see that this is even more true in this section.

Let’s look at an example.

**Example 6.** Place the expression $\sqrt{1/98}$ in simple radical form.

Sometimes it is not easy to figure out how to scale the denominator to get a perfect square, even when provided with a table of perfect squares. This is when prime factorization can come to the rescue and provide a hint. So, first express the denominator as a product of primes in exponential form: $98 = 2 \cdot 49 = 2 \cdot 7^2$.

$$\sqrt{1/98} = \sqrt{1/2 \cdot 7^2}$$

We can now easily see what is preventing the denominator from being a perfect square. The problem is the fact that not all of the exponents in the denominator are divisible by 2. We can remedy this by multiplying both numerator and denominator by 2.

$$\sqrt{1/2 \cdot 7^2} \cdot \frac{2}{2} = \sqrt{2/2^27^2}$$

Note that each prime in the denominator now has an exponent that is divisible by 2. We can now take the square root of both numerator and denominator.

$$\sqrt{2/2^27^2} = \sqrt{2/2 \cdot 7^2}$$

Take the square root of the denominator by dividing each exponent by 2.

$$\sqrt{2/2 \cdot 7^2} = \frac{\sqrt{2}}{\sqrt{2^2 \cdot 7^2}} = \frac{\sqrt{2}}{2 \cdot 7}$$

Then, of course, $2 \cdot 7 = 14$.

$$\frac{\sqrt{2}}{2 \cdot 7} = \frac{\sqrt{2}}{14}$$

In Figure 8, note how the decimal approximations of the original expression $\sqrt{1/98}$ and its simple radical form $\sqrt{2/14}$ match, strong evidence that we’ve found the correct simple radical form. That is, we cannot take a perfect square out of any radical, there are no fractions under any radical, and the denominators are clear of all radicals.

Let’s look at another example.

**Example 7.** Place the expression $12/\sqrt{54}$ in simple radical form.

Prime factor the denominator: $54 = 2 \cdot 27 = 2 \cdot 3^3$.

$$\frac{12}{\sqrt{54}} = \frac{12}{\sqrt{2 \cdot 3^3}}$$

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Figure 8. Comparing the original $\sqrt{1/98}$ with its simple radical form $\sqrt{2}/14$.

Neither prime in the denominator has an exponent divisible by 2. If we had another 2 and one more 3, then the exponents would be divisible by 2. This encourages us to multiply both numerator and denominator by $\sqrt{2} \cdot 3$.

\[
\frac{12}{\sqrt{2} \cdot 3^3} = \frac{12 \cdot \sqrt{2} \cdot 3}{\sqrt{2} \cdot 3^3} \cdot \sqrt{2} = \frac{12\sqrt{2} \cdot 3}{\sqrt{2^2} 3^4}
\]

Divide each of the exponents in the denominator by 2.

\[
\frac{12\sqrt{2} \cdot 3}{\sqrt{2^2} 3^4} = \frac{12\sqrt{2} \cdot 3}{2^1 \cdot 3^2}
\]

Then, in the numerator, $2 \cdot 3 = 6$, and in the denominator, $2 \cdot 3^2 = 18$.

\[
\frac{12\sqrt{2} \cdot 3}{2 \cdot 3^2} = \frac{12\sqrt{6}}{18}
\]

Finally, reduce to lowest terms by dividing both numerator and denominator by 6.

\[
\frac{12\sqrt{6}}{18} = \frac{2\sqrt{6}}{3}
\]

In Figure 9, the approximation for the original expression $12/\sqrt{54}$ matches that of its simple radical form $2\sqrt{6}/3$.

Figure 9. Comparing approximations of the original expression $12/\sqrt{54}$ with its simple radical form $2\sqrt{6}/3$. 
Variable Expressions

If $x$ is any real number, recall again that

$$\sqrt{x^2} = |x|.$$ 

If we combine the law of exponents for squaring a quotient with our property for taking the square root of a quotient, we can write

$$\sqrt{\left(\frac{a}{b}\right)^2} = \frac{\sqrt{a^2}}{\sqrt{b^2}}.$$ 

However, $\sqrt{(a/b)^2} = |a/b|$, while $\sqrt{a^2}/\sqrt{b^2} = |a|/|b|$. This discussion leads to the following key result.

**Quotient Rule for Absolute Value.** If $a$ and $b$ are any real numbers, then

$$\left| \frac{a}{b} \right| = \frac{|a|}{|b|},$$

provided $b \neq 0$. In words, the absolute value of a quotient is the quotient of the absolute values.

We saw this property previously in the chapter on the absolute value function, where we provided a different approach to the proof of the property. It’s interesting that we can prove this property in a completely new way using the properties of square root. We’ll see we have need for the Quotient Rule for Absolute Value in the examples that follow.

For example, if $x$ is any real number except zero, using the quotient rule for absolute value we could write

$$\left| \frac{3}{x} \right| = \frac{|3|}{|x|} = \frac{3}{|x|}.$$ 

However, there is no way to remove the absolute value bars that surround $x$ unless we know the sign of $x$. If $x > 0$ (remember, no zeros in the denominator), then $|x| = x$ and the expression becomes

$$\frac{3}{x} = \frac{3}{x}.$$ 

On the other hand, if $x < 0$, then $|x| = -x$ and the expression becomes

$$\frac{3}{|x|} = \frac{3}{-x} = -\frac{3}{x}.$$
Let’s look at another example.

Example 8. Place the expression $\sqrt{18}/x^6$ in simple radical form. Discuss the domain.

Note that $x$ cannot equal zero, otherwise the denominator of $\sqrt{18}/x^6$ would be zero, which is not allowed. However, whether $x$ is positive or negative, $x^6$ will be a positive number (raising a nonzero number to an even power always produces a positive real number), and $\sqrt{18}/x^6$ is well-defined.

Keeping in mind that $x$ is nonzero, but could either be positive or negative, we proceed by first invoking **Property 1**, taking the positive square root of both numerator and denominator of our radical expression.

\[
\sqrt{\frac{18}{x^6}} = \frac{\sqrt{18}}{\sqrt{x^6}}
\]

From the numerator, we factor a perfect square. In the denominator, we use absolute value bars to insure a positive square root.

\[
\frac{\sqrt{18}}{\sqrt{x^6}} = \frac{\sqrt{9\sqrt{2}}}{\sqrt{|x^3|}} = \frac{3\sqrt{2}}{|x^3|}
\]

We can use the Product Rule for Absolute Value to write $|x^3| = |x^2||x| = x^2|x|$. Note that we do not need to wrap $x^2$ in absolute value bars because $x^2$ is already positive.

\[
\frac{3\sqrt{2}}{|x^3|} = \frac{3\sqrt{2}}{x^2|x|}
\]

Because $x$ could be positive or negative, we cannot remove the absolute value bars around $x$. We are done.

Let’s look at another example.

Example 9. Place the expression $\sqrt{12}/x^5$ in simple radical form. Discuss the domain.

Note that $x$ cannot equal zero, otherwise the denominator of $\sqrt{12}/x^5$ would be zero, which is not allowed. Further, if $x$ is a negative number, then $x^5$ will also be a negative number (raising a negative number to an odd power produces a negative number). If $x$ were negative, then $12/x^5$ would also be negative and $\sqrt{12}/x^5$ would be undefined (you cannot take the square root of a negative number). Thus, $x$ must be a positive real number or the expression $\sqrt{12}/x^5$ is undefined.

We proceed, keeping in mind that $x$ is a positive real number. One possible approach is to first note that another factor of $x$ is needed to make the denominator a perfect square. This motivates us to multiply both numerator and denominator inside the radical by $x$.

\[
\sqrt{\frac{12}{x^5}} = \sqrt{\frac{12}{x^5} \cdot \frac{x}{x}} = \sqrt{\frac{12x}{x^6}}.
\]
We can now use Property 1 to take the square root of both numerator and denominator.

\[
\sqrt{\frac{12x}{x^6}} = \frac{\sqrt{12x}}{\sqrt{x^6}}
\]

In the numerator, we factor out a perfect square. In the denominator, absolute value bars would insure a positive square root. However, we’ve stated that \(x\) must be a positive number, so \(x^3\) is already positive and absolute value bars are not needed.

\[
\frac{\sqrt{12x}}{\sqrt{x^6}} = \frac{\sqrt{4 \cdot 3x}}{x^3} = \frac{2\sqrt{3x}}{x^3}
\]

Let’s look at another example.

**Example 10.** Given that \(x < 0\), place \(\sqrt{27/x^{10}}\) in simple radical form.

One possible approach would be to factor out a perfect square and write

\[
\sqrt{\frac{27}{x^{10}}} = \sqrt{\frac{9}{x^{10}}} \cdot \sqrt{3} = \sqrt{\left(\frac{3}{x^5}\right)^2 \cdot \sqrt{3}} = \frac{3}{x^5} \cdot \sqrt{3}.
\]

Now, \(|3/x^5| = |3|/(|x^4||x|) = 3/(x^4|x|)\), since \(x^4 > 0\). Thus,

\[
\left|\frac{3}{x^5}\right| \cdot \sqrt{3} = \frac{3}{x^4|x|} \cdot \sqrt{3}.
\]

However, we are given that \(x < 0\), so \(|x| = -x\) and we can write

\[
\frac{3}{x^4|x|} \cdot \sqrt{3} = \frac{3}{(x^4)(-x)} \cdot \sqrt{3} = -\frac{3}{x^5} \cdot \sqrt{3}.
\]

We can move \(\sqrt{3}\) into the numerator and write

\[
-\frac{3}{x^5} \cdot \sqrt{3} = -\frac{3\sqrt{3}}{x^5}.
\]

Again, it’s instructive to test the validity of this result using your graphing calculator. Supposedly, the result is true for all values of \(x < 0\). So, store \(-1\) in \(x\), then enter the original expression and its simple radical form, then compare the approximations, as shown in Figures 10(a), (b), and (c).

<table>
<thead>
<tr>
<th>Store (-1) in (x).</th>
<th>Approximate (\sqrt{27/x^{10}}).</th>
<th>Approximate (-3\sqrt{3}/x^5).</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Graph 1" /></td>
<td><img src="image2.png" alt="Graph 2" /></td>
<td><img src="image3.png" alt="Graph 3" /></td>
</tr>
</tbody>
</table>

**Figure 10.** Comparing the original expression and its simple radical form at \(x = -1\).
**Alternative approach.** A slightly different approach would again begin by taking the square root of both numerator and denominator.

\[
\sqrt{\frac{27}{x^{10}}} = \frac{\sqrt{27}}{\sqrt{x^{10}}}
\]

Now, \(\sqrt{27} = \sqrt{9}\sqrt{3} = 3\sqrt{3}\) and we insure that \(\sqrt{x^{10}}\) produces a positive number by using absolute value bars. That is, \(\sqrt{x^{10}} = |x^5|\) and

\[
\frac{\sqrt{27}}{\sqrt{x^{10}}} = \frac{3\sqrt{3}}{|x^5|}.
\]

However, using the product rule for absolute value and the fact that \(x^4 > 0\), \(|x^5| = |x^4|x = x^4|x|\) and

\[
\frac{3\sqrt{3}}{|x^5|} = \frac{3\sqrt{3}}{x^4|x|}.
\]

Finally, we are given that \(x < 0\), so \(|x| = -x\) and we can write

\[
\frac{3\sqrt{3}}{x^4|x|} = \frac{3\sqrt{3}}{(x^4)(-x)} = -\frac{3\sqrt{3}}{x^5}.
\]

Note that the simple radical form (12) of our alternative approach matches perfectly the simple radical form (11) of our first approach.
9.3 Exercises

1. Use a calculator to first approximate \( \sqrt{5}/\sqrt{2} \). On the same screen, approximate \( \sqrt{5}/2 \). Report the results on your homework paper.

2. Use a calculator to first approximate \( \sqrt{7}/\sqrt{5} \). On the same screen, approximate \( \sqrt{7}/5 \). Report the results on your homework paper.

3. Use a calculator to first approximate \( \sqrt{12}/\sqrt{2} \). On the same screen, approximate \( \sqrt{6} \). Report the results on your homework paper.

4. Use a calculator to first approximate \( \sqrt{15}/\sqrt{5} \). On the same screen, approximate \( \sqrt{3} \). Report the results on your homework paper.

In Exercises 5-16, place each radical expression in simple radical form. As in Example 2 in the narrative, check your result with your calculator.

5. \( \sqrt{3}/8 \)
6. \( \sqrt{5}/12 \)
7. \( \sqrt{11}/20 \)
8. \( \sqrt{3}/2 \)
9. \( \sqrt{11}/18 \)
10. \( \sqrt{7}/5 \)
11. \( \sqrt{4}/3 \)
12. \( \sqrt{16}/5 \)
13. \( \sqrt{49}/12 \)
14. \( \sqrt{81}/20 \)
15. \( \sqrt{100}/7 \)
16. \( \sqrt{36}/5 \)

In Exercises 17-28, place each radical expression in simple radical form. As in Example 4 in the narrative, check your result with your calculator.

17. \( 1/\sqrt{12} \)
18. \( 1/\sqrt{8} \)
19. \( 1/\sqrt{20} \)
20. \( 1/\sqrt{27} \)
21. \( 6/\sqrt{8} \)
22. \( 4/\sqrt{12} \)

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23. \( \frac{5}{\sqrt{20}} \)

24. \( \frac{9}{\sqrt{27}} \)

25. \( \frac{6}{2\sqrt{3}} \)

26. \( \frac{10}{3\sqrt{5}} \)

27. \( \frac{15}{2\sqrt{20}} \)

28. \( \frac{3}{2\sqrt{18}} \)

In Exercises 29-36, place the given radical expression in simple form. Use prime factorization as in Example 8 in the narrative to help you with the calculations. As in Example 6, check your result with your calculator.

29. \( \frac{1}{\sqrt{96}} \)

30. \( \frac{1}{\sqrt{432}} \)

31. \( \frac{1}{\sqrt{250}} \)

32. \( \frac{1}{\sqrt{108}} \)

33. \( \sqrt{\frac{5}{96}} \)

34. \( \sqrt{\frac{2}{135}} \)

35. \( \sqrt{\frac{2}{1485}} \)

36. \( \sqrt{\frac{3}{280}} \)

In Exercises 37-44, place each of the given radical expressions in simple radical form. Make no assumptions about the sign of any variable. Variables can represent either positive or negative numbers.

37. \( \sqrt{\frac{8}{x^4}} \)

38. \( \sqrt{\frac{12}{x^6}} \)

39. \( \sqrt{\frac{20}{x^2}} \)

40. \( \sqrt{\frac{32}{x^{14}}} \)

41. \( \frac{2}{\sqrt{8x^5}} \)

42. \( \frac{3}{\sqrt{12x^6}} \)

43. \( \frac{10}{\sqrt{20x^{10}}} \)

44. \( \frac{12}{\sqrt{6x^4}} \)

In Exercises 45-48, follow the lead of Example 8 in the narrative to craft a solution.

45. Given that \( x < 0 \), place the radical expression \( 6/\sqrt{2x^6} \) in simple radical form. Check your solution on your calculator for \( x = -1 \).

46. Given that \( x > 0 \), place the radical expression \( 4/\sqrt{12x^3} \) in simple radical form. Check your solution on your calculator for \( x = 1 \).
47. Given that \(x > 0\), place the radical expression \(\frac{8}{\sqrt{8x^5}}\) in simple radical form. Check your solution on your calculator for \(x = 1\).

48. Given that \(x < 0\), place the radical expression \(\frac{15}{\sqrt{20x^6}}\) in simple radical form. Check your solution on your calculator for \(x = -1\).

In **Exercises 49-56**, place each of the radical expressions in simple form. Assume that all variables represent positive numbers.

49. \(\sqrt{\frac{12}{x}}\)

50. \(\sqrt{\frac{18}{x}}\)

51. \(\sqrt{\frac{50}{x^3}}\)

52. \(\sqrt{\frac{72}{x^5}}\)

53. \(\frac{1}{\sqrt{50x}}\)

54. \(\frac{2}{\sqrt{18x}}\)

55. \(\frac{3}{\sqrt{27x^3}}\)

56. \(\frac{5}{\sqrt{10x^5}}\)
9.3 Answers

1. \(\sqrt[5]{5}/\sqrt[2]{2}\)  
31. \(\sqrt{10}/50\)

33. \(\sqrt{30}/24\)

35. \(\sqrt{330}/495\)

37. \(2\sqrt{2}/x^2\)

39. \(2\sqrt{5}/|x|\)

41. \(\sqrt{2}/(2x^4)\)

43. \(\sqrt{5}/(x^4|x|)\)

45. \(-3\sqrt{2}/x^3\)

47. \(2\sqrt{2x}/x^3\)

49. \(2\sqrt{3x}/x\)

51. \(5\sqrt{2x}/x^2\)

53. \(\sqrt{2x}/(10x)\)

55. \(\sqrt{3x}/(3x^2)\)

5. \(\sqrt{6}/4\)

7. \(\sqrt{55}/10\)

9. \(\sqrt{22}/6\)

11. \(2\sqrt{3}/3\)

13. \(7\sqrt{3}/6\)

15. \(10\sqrt{7}/7\)

17. \(\sqrt{3}/6\)

19. \(\sqrt{5}/10\)

21. \(3\sqrt{2}/2\)

23. \(\sqrt{5}/2\)

25. \(\sqrt{3}\)

27. \(3\sqrt{5}/4\)

29. \(\sqrt{6}/24\)
9.4 Radical Expressions

In the previous two sections, we learned how to multiply and divide square roots. Specifically, we are now armed with the following two properties.

**Property 1.** Let $a$ and $b$ be any two real nonnegative numbers. Then,

$$\sqrt{a}\sqrt{b} = \sqrt{ab},$$

and, provided $b \neq 0$,

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}.$$

In this section, we will simplify a number of more extensive expressions containing square roots, particularly those that are fundamental to your work in future mathematics courses.

Let’s begin by building some fundamental skills.

**The Associative Property**

We recall the associative property of multiplication.

**Associative Property of Multiplication.** Let $a$, $b$, and $c$ be any real numbers. The associative property of multiplication states that

$$(ab)c = a(bc).$$

(2)

Note that the order of the numbers on each side of equation (2) has not changed. The numbers on each side of the equation are in the order $a$, $b$, and then $c$.

However, the grouping has changed. On the left, the parentheses around the product of $a$ and $b$ instruct us to perform that product first, then multiply the result by $c$. On the right, the grouping is different; the parentheses around $b$ and $c$ instruct us to perform that product first, then multiply by $a$. The key point to understand is the fact that the different groupings make no difference. We get the same answer in either case.

For example, consider the product $2 \cdot 3 \cdot 4$. If we multiply 2 and 3 first, then multiply the result by 4, we get

$$(2 \cdot 3) \cdot 4 = 6 \cdot 4 = 24.$$ 

On the other hand, if we multiply 3 and 4 first, then multiply the result by 2, we get

$$2 \cdot (3 \cdot 4) = 2 \cdot 12 = 24.$$
Note that we get the same result in either case. That is,

\[(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4).\]

The associative property, seemingly trivial, takes on an extra level of sophistication if we apply it to expressions containing radicals. Let’s look at an example.

**Example 3.** Simplify the expression \(3(2\sqrt{5})\). Place your answer in simple radical form.

Currently, the parentheses around 2 and \(\sqrt{5}\) require that we multiply those two numbers first. However, the associative property of multiplication allows us to regroup, placing the parentheses around 3 and 2, multiplying those two numbers first, then multiplying the result by \(\sqrt{5}\). We arrange the work as follows.

\[3(2\sqrt{5}) = (3 \cdot 2)\sqrt{5} = 6\sqrt{5}.\]

Readers should note the similarity to a very familiar manipulation.

\[3(2x) = (3 \cdot 2)x = 6x\]

In practice, when we became confident with this regrouping, we began to skip the intermediate step and simply state that \(3(2x) = 6x\). In a similar vein, once you become confident with regrouping, you should simply state that \(3(2\sqrt{5}) = 6\sqrt{5}\). If called upon to explain your answer, you must be ready to explain how you regrouped according to the associative property of multiplication. Similarly,

\[-4(5\sqrt{7}) = -20\sqrt{7}, \quad 12(5\sqrt{11}) = 60\sqrt{11}, \quad \text{and} \quad -5(-3\sqrt{3}) = 15\sqrt{3}.\]

**The Commutative Property of Multiplication**

We recall the commutative property of multiplication.

**Commutative Property of Multiplication.** Let \(a\) and \(b\) be any real numbers. The *commutative property of multiplication* states that

\[ab = ba.\]  \hspace{1cm} (4)

The commutative property states that the order of multiplication is irrelevant. For example, \(2 \cdot 3\) is the same as \(3 \cdot 2\); they both equal 6. This seemingly trivial property, coupled with the associative property of multiplication, allows us to change the order of multiplication and regroup as we please.
Example 5. Simplify the expression $\sqrt{5}(2\sqrt{3})$. Place your answer in simple radical form.

What we’d really like to do is first multiply $\sqrt{5}$ and $\sqrt{3}$. In order to do this, we must first regroup, then switch the order of multiplication as follows.

$$\sqrt{5}(2\sqrt{3}) = (\sqrt{5} \cdot 2)\sqrt{3} = (2\sqrt{5})\sqrt{3}$$

This is allowed by the associative and commutative properties of multiplication. Now, we regroup again and multiply.

$$(2\sqrt{5})\sqrt{3} = 2(\sqrt{5}\sqrt{3}) = 2\sqrt{15}$$

In practice, this is far too much work for such a simple calculation. Once we understand the associative and commutative properties of multiplication, the expression $a \cdot b \cdot c$ is unambiguous. Parentheses are not needed. We know that we can change the order of multiplication and regroup as we please. Therefore, when presented with the product of three numbers, simply multiply two of your choice together, then multiply the result by the third remaining number.

In the case of $\sqrt{5}(2\sqrt{3})$, we choose to first multiply $\sqrt{5}$ and $\sqrt{3}$, which is $\sqrt{15}$, then multiply this result by 2 to get $2\sqrt{15}$. Similarly,

$$\sqrt{5}(2\sqrt{7}) = 2\sqrt{35} \quad \text{and} \quad \sqrt{x}(3\sqrt{5}) = 3\sqrt{5x}.$$ 

Example 6. Simplify the expression $\sqrt{6}(4\sqrt{8})$. Place your answer in simple radical form.

We start by multiplying $\sqrt{6}$ and $\sqrt{8}$, then the result by 4.

$$\sqrt{6}(4\sqrt{8}) = 4\sqrt{48}$$

Now, $48 = 16 \cdot 3$, so we can extract a perfect square.

$$4\sqrt{48} = 4(\sqrt{16\sqrt{3}}) = 4(4\sqrt{3})$$

Again, we choose to multiply the fours, then the result by the square root of three. That is,

$$4(4\sqrt{3}) = 16\sqrt{3}.$$ 

By induction, we can argue that the associative and commutative properties will allow us to group and arrange the product of more than three numbers in any order that we please.
Example 7. Simplify the expression \( (2\sqrt{12})(3\sqrt{3}) \). Place your answer in simple radical form.

We’ll first take the product of 2 and 3, then the product of \( \sqrt{12} \) and \( \sqrt{3} \), then multiply these results together.

\[
(2\sqrt{12})(3\sqrt{3}) = (2 \cdot 3)(\sqrt{12\sqrt{3}}) = 6\sqrt{36}
\]

Of course, \( \sqrt{36} = 6 \), so we can simplify further.

\[
6\sqrt{36} = 6 \cdot 6 = 36
\]

The Distributive Property

Recall the distributive property for real numbers.

**Distributive Property.** Let \( a, b, \) and \( c \) be any real numbers. Then,

\[
a(b + c) = ab + ac.
\]

You might recall the following operation, where you “distribute the 2,” multiplying each term in the parentheses by 2.

\[
2(3 + x) = 6 + 2x
\]

You can do precisely the same thing with radical expressions.

\[
2(3 + \sqrt{5}) = 6 + 2\sqrt{5}
\]

Like the familiar example above, we “distributed the 2,” multiplying each term in the parentheses by 2.

Let’s look at more examples.

Example 9. Use the distributive property to expand the expression \( \sqrt{12}(3 + \sqrt{3}) \), placing your final answer in simple radical form.

First, distribute the \( \sqrt{12} \), multiplying each term in the parentheses by \( \sqrt{12} \). Note that \( \sqrt{12}\sqrt{3} = \sqrt{36} \).

\[
\sqrt{12}(3 + \sqrt{3}) = 3\sqrt{12} + \sqrt{36} = 3\sqrt{12} + 6
\]
However, this last expression is not in simple radical form, as we can factor out a perfect square ($12 = 4 \cdot 3$).

\[ 3\sqrt{12} + 6 = 3(\sqrt{4\sqrt{3}}) + 6 \]
\[ = 3(2\sqrt{3}) + 6 \]
\[ = 6\sqrt{3} + 6 \]

It doesn’t matter whether the monomial factor is in the front or rear of the sum, you still distribute the monomial times each term in the parentheses.

\begin{itemize}
  \item \textbf{Example 10.} \textit{Use the distributive property to expand $(\sqrt{3} + 2\sqrt{2})\sqrt{6}$. Place your answer in simple radical form.}
\end{itemize}

First, multiply each term in the parentheses by $\sqrt{6}$.

\[ (\sqrt{3} + 2\sqrt{2})\sqrt{6} = \sqrt{18} + 2\sqrt{12} \]

To obtain the second term of this result, we chose to first multiply $\sqrt{2}$ and $\sqrt{6}$, which is $\sqrt{12}$, then we multiplied this result by 2. Now, we can factor perfect squares from both 18 and 12.

\[ \sqrt{18} + 2\sqrt{12} = \sqrt{9\sqrt{2}} + 2(\sqrt{4\sqrt{3}}) \]
\[ = 3\sqrt{2} + 2(2\sqrt{3}) \]
\[ = 3\sqrt{2} + 4\sqrt{3} \]

Remember, you can check your results with your calculator. In Figure 1(a), we’ve found a decimal approximation for the original expression $(\sqrt{3} + 2\sqrt{2})\sqrt{6}$, and in Figure 1(b) we have a decimal approximation for our solution $3\sqrt{2} + 4\sqrt{3}$. Note that they are the same, providing evidence that our solution is correct.

\begin{figure}[h]
\centering
\begin{tabular}{c c}
\includegraphics[width=0.4\textwidth]{a.png} & \includegraphics[width=0.4\textwidth]{b.png} \\
(a) Approximating $(\sqrt{3} + 2\sqrt{2})\sqrt{6}$. & (b) Approximating $3\sqrt{2} + 4\sqrt{3}$. \\
\end{tabular}
\caption{Comparing the original expression with its simple radical form.}
\end{figure}

The distributive property is also responsible in helping us combine “like terms.” For example, you might remember that $3x + 5x = 8x$, a seemingly simple calculation, but
it is the distributive property that actually provides this solution. Note how we use the

distributive property to factor $x$ from each term.

$$3x + 5x = (3 + 5)x$$

Hence, $3x + 5x = 8x$. You can do the same thing with radical expressions.

$$3\sqrt{2} + 5\sqrt{2} = (3 + 5)\sqrt{2}$$

Hence, $3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$, and the structure of this result is identical to that shown
in $3x + 5x = 8x$. There is no difference in the way we combine these “like terms.” We
repeat the common factor and add coefficients. For example,

$$2\sqrt{3} + 9\sqrt{3} = 11\sqrt{3}, \quad -4\sqrt{2} + 2\sqrt{2} = -2\sqrt{2}, \quad \text{and} \quad -3x\sqrt{x} + 5x\sqrt{x} = 2x\sqrt{x}.$$  

In each case above, we’re adding “like terms,” by repeating the common factor and
adding coefficients.

In the case that we don’t have like terms, as in $3x + 5y$, there is nothing to be
done. In like manner, each of the following expressions have no like terms that you can
combine. They are as simplified as they are going to get.

$$3\sqrt{2} + 5\sqrt{3}, \quad 2\sqrt{11} - 8\sqrt{10}, \quad \text{and} \quad 2\sqrt{x} + 5\sqrt{y}$$

However, there are times when it can look as if you don’t have like terms, but when
you place everything in simple radical form, you discover that you do have like terms
that can be combined by adding coefficients.

► Example 11. Simplify the expression $5\sqrt{27} + 8\sqrt{3}$, placing the final expression in
simple radical form.

We can extract a perfect square ($27 = 9 \cdot 3$).

$$5\sqrt{27} + 8\sqrt{3} = 5(\sqrt{9}\sqrt{3}) + 8\sqrt{3} = 5(3\sqrt{3}) + 8\sqrt{3} = 15\sqrt{3} + 8\sqrt{3}$$

Note that we now have “like terms” that can be combined by adding coefficients.

$$15\sqrt{3} + 8\sqrt{3} = 23\sqrt{3}$$

A comparison of the original expression and its simplified form is shown in Figures 2(a)
and (b).

► Example 12. Simplify the expression $2\sqrt{20} + \sqrt{8} + 3\sqrt{5} + 4\sqrt{2}$, placing the result
in simple radical form.

We can extract perfect squares ($20 = 4 \cdot 5$ and $8 = 4 \cdot 2$).
(a) Approximating $5\sqrt{27} + 8\sqrt{3}$.

(b) Approximating $23\sqrt{3}$.

Figure 2. Comparing the original expression with its simplified form.

$$2\sqrt{20} + \sqrt{8} + 3\sqrt{5} + 4\sqrt{2} = 2(\sqrt{4}\sqrt{5}) + \sqrt{4\sqrt{2}} + 3\sqrt{5} + 4\sqrt{2}$$
$$= 2(2\sqrt{5}) + 2\sqrt{2} + 3\sqrt{5} + 4\sqrt{2}$$
$$= 4\sqrt{5} + 2\sqrt{2} + 3\sqrt{5} + 4\sqrt{2}$$

Now we can combine like terms by adding coefficients.

$$4\sqrt{5} + 2\sqrt{2} + 3\sqrt{5} + 4\sqrt{2} = 7\sqrt{5} + 6\sqrt{2}$$

Fractions can be a little tricky.

Example 13. Simplify $\sqrt{27} + \frac{1}{\sqrt{12}}$, placing the result in simple radical form.

We can extract a perfect square root ($27 = 9 \cdot 3$). The denominator in the second term is $12 = 2^2 \cdot 3$, so one more 3 is needed in the denominator to make a perfect square.

$$\sqrt{27} + \frac{1}{\sqrt{12}} = \sqrt{9\sqrt{3}} + \frac{1}{\sqrt{12}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$
$$= 3\sqrt{3} + \frac{\sqrt{3}}{\sqrt{36}}$$
$$= 3\sqrt{3} + \frac{\sqrt{3}}{6}$$

To add these fractions, we need a common denominator of 6.

$$3\sqrt{3} + \frac{\sqrt{3}}{6} = \frac{3\sqrt{3}}{1} \cdot \frac{6}{6} + \frac{\sqrt{3}}{6}$$
$$= \frac{18\sqrt{3}}{6} + \frac{\sqrt{3}}{6}$$

We can now combine numerators by adding coefficients.

$$\frac{18\sqrt{3}}{6} + \frac{\sqrt{3}}{6} = \frac{19\sqrt{3}}{6}$$

Decimal approximations of the original expression and its simplified form are shown in Figures 3(a) and (b).
Chapter 9 Radical Functions

(a) Approximating $\sqrt{27} + 1/\sqrt{12}$.

(b) Approximating $19\sqrt{3}/6$.

Figure 3. Comparing the original expression and its simple radical form.

At first glance, the lack of a monomial in the product $(x + 1)(x + 3)$ makes one think that the distributive property will not help us find the product. However, if we think of the second factor as a single unit, we can distribute it times each term in the first factor.

$$(x + 1)(x + 3) = x(x + 3) + 1(x + 3)$$

Apply the distributive property a second time, then combine like terms.

$$x(x + 3) + 1(x + 3) = x^2 + 3x + x + 3$$

$$= x^2 + 4x + 3$$

We can handle products with radical expressions in the same manner.

Example 14. Simplify $(2 + \sqrt{2})(3 + 5\sqrt{2})$. Place your result in simple radical form.

Think of the second factor as a single unit and distribute it times each term in the first factor.

$$(2 + \sqrt{2})(3 + 5\sqrt{2}) = 2(3 + 5\sqrt{2}) + \sqrt{2}(3 + 5\sqrt{2})$$

Now, use the distributive property again.

$$2(3 + 5\sqrt{2}) + \sqrt{2}(3 + 5\sqrt{2}) = 6 + 10\sqrt{2} + 3\sqrt{2} + 5\sqrt{4}$$

Note that in finding the last term, $\sqrt{2}\sqrt{2} = \sqrt{4}$. Now, $\sqrt{4} = 2$, then we can combine like terms.

$$6 + 10\sqrt{2} + 3\sqrt{2} + 5(2) = 6 + 10\sqrt{2} + 3\sqrt{2} + 10$$

$$= 16 + 13\sqrt{2}$$

Decimal approximations of the original expression and its simple radical form are shown in Figures 4(a) and (b).
(a) Approximating $(2 + \sqrt{2})(3 + 5\sqrt{2})$.

(b) Approximating $16 + 13\sqrt{2}$.

Figure 4. Comparing the original expression with its simple radical form.

**Special Products**

There are two special products that have important applications involving radical expressions, perhaps one more than the other. The first is the well-known difference of two squares pattern.

**Difference of Squares.** Let $a$ and $b$ be any numbers. Then,

$$(a + b)(a - b) = a^2 - b^2.$$

This pattern involves two binomial factors having identical first and second terms, the terms in one factor separated by a plus sign, the terms in the other factor separated by a minus sign. When we see this pattern of multiplication, we should square the first term of either factor, square the second term, then subtract the results. For example,

$$(2x + 3)(2x - 3) = 4x^2 - 9.$$  

This special product applies equally well when the first and/or second terms involve radical expressions.

**Example 15.** Use the difference of squares pattern to multiply $(2 + \sqrt{11})(2 - \sqrt{11})$.

Note that this multiplication has the form $(a + b)(a - b)$, so we apply the difference of squares pattern to get

$$(2 + \sqrt{11})(2 - \sqrt{11}) = (2)^2 - (\sqrt{11})^2.$$

Of course, $2^2 = 4$ and $(\sqrt{11})^2 = 11$, so we can finish as follows.

$$(2)^2 - (\sqrt{11})^2 = 4 - 11 = -7$$
Example 16. Use the difference of squares pattern to multiply \((2\sqrt{5}+3\sqrt{7})(2\sqrt{5}-3\sqrt{7})\).

Again, this product has the special form \((a + b)(a - b)\), so we apply the difference of squares pattern to get

\[(2\sqrt{5} + 3\sqrt{7})(2\sqrt{5} - 3\sqrt{7}) = (2\sqrt{5})^2 - (3\sqrt{7})^2.\]

Next, we square a product of two factors according to the rule \((ab)^2 = a^2b^2\). Thus,

\[(2\sqrt{5})^2 = (2)^2(\sqrt{5})^2 = 4 \cdot 5 = 20\]

and

\[(3\sqrt{7})^2 = (3)^2(\sqrt{7})^2 = 9 \cdot 7 = 63.\]

Thus, we can complete the multiplication \((2\sqrt{5} + 3\sqrt{7})(2\sqrt{5} - 3\sqrt{7})\) with

\[(2\sqrt{5})^2 - (3\sqrt{7})^2 = 20 - 63 = -43.\]

This result is easily verified with a calculator, as shown in Figure 5.

![Figure 5. Approximating \((2\sqrt{5} + 3\sqrt{7})(2\sqrt{5} - 3\sqrt{7})\).](image)

The second pattern of interest is the shortcut for squaring a binomial.

**Squaring a Binomial.** Let \(a\) and \(b\) be numbers. Then,

\[(a + b)^2 = a^2 + 2ab + b^2.\]

Here we square the first and second terms of the binomial, then produce the middle term of the result by multiplying the first and second terms and doubling that result. For example,

\[(2x + 9)^2 = (2x)^2 + 2(2x)(9) + (9)^2 = 4x^2 + 36x + 81.\]

This pattern can also be applied to binomials containing radical expressions.
Example 17. Use the squaring a binomial pattern to expand \((2\sqrt{x} + \sqrt{5})^2\). Place your result in simple radical form. Assume that \(x\) is a positive real number \((x > 0)\).

Applying the squaring a binomial pattern, we get
\[
(2\sqrt{x} + \sqrt{5})^2 = (2\sqrt{x})^2 + 2(2\sqrt{x})(\sqrt{5}) + (\sqrt{5})^2.
\]
As before, \((2\sqrt{x})^2 = (2)^2(\sqrt{x})^2 = 4x\) and \((\sqrt{5})^2 = 5\). In the case of \(2(2\sqrt{x})(\sqrt{5})\), note that we are multiplying four numbers together. The associative and commutative properties state that we can multiply these four numbers in any order that we please. So, the product of 2 and 2 is 4, the product of \(\sqrt{x}\) and \(\sqrt{5}\) is \(\sqrt{5x}\), then we multiply these results to produce the result \(4\sqrt{5x}\). Thus,
\[
(2\sqrt{x})^2 + 2(2\sqrt{x})(\sqrt{5}) + (\sqrt{5})^2 = 4x + 4\sqrt{5x} + 5.
\]

Rationalizing Denominators

As we saw in the previous section, the instruction “rationalize the denominator” is a request to remove all radical expressions from the denominator. Of course, this is the “third guideline of simple radical form,” but there are times, particularly in calculus, when the instruction changes to “rationalize the numerator.” Of course, this is a request to remove all radicals from the numerator.

You can’t have both worlds. You can either remove radical expressions from the denominator or from the numerator, but not both. If no instruction is given, assume that the “third guideline of simple radical form” is in play and remove all radical expressions from the denominator. We’ve already done a little of this in previous sections, but here we address a slightly more complicated type of expression.

Example 18. In the expression
\[
\frac{3}{2 + \sqrt{2}},
\]
rationalize the denominator.

The secret lies in the difference of squares pattern, \((a + b)(a - b) = a^2 - b^2\). For example,
\[
(2 + \sqrt{2})(2 - \sqrt{2}) = (2)^2 - (\sqrt{2})^2 = 4 - 2 = 2.
\]
This provides a terrific hint at how to proceed with rationalizing the denominator of the expression \(3/(2 + \sqrt{2})\). Multiply both numerator and denominator by \(2 - \sqrt{2}\).
\[
\frac{3}{2 + \sqrt{2}} = \frac{3}{2 + \sqrt{2}} \cdot \frac{2 - \sqrt{2}}{2 - \sqrt{2}} = \frac{3(2 - \sqrt{2})}{(2 + \sqrt{2})(2 - \sqrt{2})} = \frac{6 - 3\sqrt{2}}{4}.
\]
Multiply numerators and denominators.

\[
\frac{3}{2 + \sqrt{2}} \cdot \frac{2 - \sqrt{2}}{2 - \sqrt{2}} = \frac{3(2 - \sqrt{2})}{(2 + \sqrt{2})(2 - \sqrt{2})} = \frac{6 - 3\sqrt{2}}{(2)^2 - (\sqrt{2})^2} = \frac{6 - 3\sqrt{2}}{4 - 2} = \frac{6 - 3\sqrt{2}}{2}
\]

Note that it is tempting to cancel the 2 in the denominator into the 6 in the numerator, but you are not allowed to cancel terms that are separated by a minus sign. This is a common error, so don’t fall prey to this mistake.

In Figures 6(a) and (b), we compare decimal approximations of the original expression and its simple radical form.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figures.png}
\caption{Comparing the original expression with its simple radical form.}
\end{figure}

\begin{itemize}
\item \textbf{Example 19.} In the expression

\[
\frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}},
\]

\textit{rationalize the denominator.}

Multiply numerator and denominator by \(\sqrt{3} + \sqrt{2}\).

\[
\frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}} = \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}} \cdot \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2}} = \frac{(\sqrt{3} + \sqrt{2})^2}{(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})}
\]

\end{itemize}
In the denominator, we have the difference of two squares. Thus,

$$(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2}) = (\sqrt{3})^2 - (\sqrt{2})^2 = 3 - 2 = 1.$$  

Note that this clears the denominator of radicals. This is the reason we multiply numerator and denominator by $\sqrt{3} + \sqrt{2}$. In the numerator, we can use the squaring a binomial shortcut to multiply.

$$(\sqrt{3} + \sqrt{2})^2 = (\sqrt{3})^2 + 2(\sqrt{3})(\sqrt{2}) + (\sqrt{2})^2$$  
$$= 3 + 2\sqrt{6} + 2$$  
$$= 5 + 2\sqrt{6}$$

Thus, we can complete the simplification started above.

$$\frac{(\sqrt{3} + \sqrt{2})^2}{(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})} = \frac{5 + 2\sqrt{6}}{1} = 5 + 2\sqrt{6}$$

In Figures 7(a) and (b), we compare the decimal approximations of the original expression with its simple radical form.

Revisiting the Quadratic Formula

We can use what we’ve learned to place solutions provided by the quadratic formula in simple form. Let’s look at an example.

**Example 20.** Solve the equation $x^2 = 2x + 2$ for $x$. Place your solution in simple radical form.

The equation is nonlinear, so make one side zero.

$$x^2 - 2x - 2 = 0$$

Compare this result with the general form $ax^2 + bx + c = 0$ and note that $a = 1, b = -2$ and $c = -2$. Write down the quadratic formula, make the substitutions, then simplify.
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} = \frac{2 \pm \sqrt{12}}{2} \]

Note that we can factor a perfect square from the radical in the numerator.

\[ x = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm \sqrt{4\sqrt{3}}}{2} = \frac{2 \pm 2\sqrt{3}}{2} \]

At this point, note that both numerator and denominator are divisible by 2. There are several ways that we can proceed with the reduction.

- Some people prefer to factor, then cancel.
  \[ x = \frac{2 \pm 2\sqrt{3}}{2} = \frac{2(1 \pm \sqrt{3})}{2} = \frac{2(1 \pm \sqrt{3})}{2} = 1 \pm \sqrt{3} \]

- Some prefer to use the distributive property.
  \[ x = \frac{2 \pm 2\sqrt{3}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3} \]

In each case, the final form of the answer is in simple radical form and it is reduced to lowest terms.

**Warning 21.** When working with the quadratic formula, one of the most common algebra mistakes is to cancel addends instead of factors, as in

\[ \frac{2 \pm 2\sqrt{3}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = \pm 2\sqrt{3}. \]

Please try to avoid making this mistake.

Let’s look at another example.

- **Example 22.** Solve the equation \(3x^2 - 2x = 6\) for \(x\). Place your solution in simple radical form.

  This equation is nonlinear. Move every term to one side of the equation, making the other side of the equation equal to zero.

  \[ 3x^2 - 2x - 6 = 0 \]

  Compare with the general form \(ax^2 + bx + c = 0\) and note that \(a = 3\), \(b = -2\), and \(c = -6\). Write down the quadratic formula and substitute.

  \[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(2) \pm \sqrt{(-2)^2 - 4(3)(-6)}}{2(3)} = \frac{2 \pm \sqrt{76}}{6} \]

  Factor a perfect square from the radical in the numerator.
We choose to factor and cancel.

\[
x = \frac{2 \pm \sqrt{76}}{6} = \frac{2 \pm \sqrt{4\sqrt{19}}}{6} = \frac{2 \pm 2\sqrt{19}}{6}
\]

\[
x = \frac{2(1 \pm \sqrt{19})}{2 \cdot 3} = \frac{2(1 \pm \sqrt{19})}{2 \cdot 3} = \frac{1 \pm \sqrt{19}}{3}
\]
9.4 Exercises

In Exercises 1–14, place each of the radical expressions in simple radical form. Check your answer with your calculator.

1. \(2(5\sqrt{7})\)
2. \(-3(2\sqrt{3})\)
3. \(-\sqrt{3}(2\sqrt{5})\)
4. \(\sqrt{2}(3\sqrt{7})\)
5. \(\sqrt{3}(5\sqrt{6})\)
6. \(\sqrt{2}(-3\sqrt{10})\)
7. \((2\sqrt{5})(-3\sqrt{3})\)
8. \((-5\sqrt{2})(-2\sqrt{7})\)
9. \((-4\sqrt{3})(2\sqrt{6})\)
10. \((2\sqrt{5})(-3\sqrt{10})\)
11. \((2\sqrt{3})^2\)
12. \((-3\sqrt{5})^2\)
13. \((-5\sqrt{2})^2\)
14. \((7\sqrt{11})^2\)
15. \(2(3 + \sqrt{5})\)
16. \(-3(4 - \sqrt{7})\)
17. \(2(-5 + 4\sqrt{2})\)
18. \(-3(4 - 3\sqrt{2})\)
19. \(\sqrt{2}(2 + \sqrt{2})\)
20. \(\sqrt{3}(4 - \sqrt{6})\)
21. \(\sqrt{2}(\sqrt{10} + \sqrt{14})\)
22. \(\sqrt{3}(\sqrt{15} - \sqrt{33})\)

In Exercises 15–22, use the distributive property to multiply. Place your final answer in simple radical form. Check your result with your calculator.

23. \(-5\sqrt{2} + 7\sqrt{2}\)
24. \(2\sqrt{3} + 3\sqrt{3}\)
25. \(2\sqrt{6} - 8\sqrt{6}\)
26. \(\sqrt{7} - 3\sqrt{7}\)
27. \(2\sqrt{3} - 4\sqrt{2} + 3\sqrt{3}\)
28. \(7\sqrt{5} + 2\sqrt{7} - 3\sqrt{5}\)
29. \(2\sqrt{3} + 5\sqrt{2} - 7\sqrt{3} + 2\sqrt{2}\)
30. \(3\sqrt{11} - 2\sqrt{7} - 2\sqrt{11} + 4\sqrt{7}\)

In Exercises 31–40, combine like terms where possible. Place your final answer in simple radical form. Use your calculator to check your result.

31. \(\sqrt{45} + \sqrt{20}\)
32. \(-4\sqrt{45} - 4\sqrt{20}\)
33. \(2\sqrt{18} - \sqrt{8}\)

\[\text{Version: Fall 2007}\]
34. $-\sqrt{20} + 4\sqrt{45}$
35. $-5\sqrt{27} + 5\sqrt{12}$
36. $3\sqrt{T^2} - 2\sqrt{27}$
37. $4\sqrt{20} + 4\sqrt{45}$
38. $-2\sqrt{18} - 5\sqrt{8}$
39. $2\sqrt{45} + 5\sqrt{20}$
40. $3\sqrt{27} - 4\sqrt{12}$

In Exercises 41-48, simplify each of the given rational expressions. Place your final answer in simple radical form. Check your result with your calculator.

41. $\sqrt{2} - \frac{1}{\sqrt{2}}$
42. $3\sqrt{3} - \frac{3}{\sqrt{3}}$
43. $2\sqrt{2} - \frac{2}{\sqrt{2}}$
44. $4\sqrt{5} - \frac{5}{\sqrt{5}}$
45. $5\sqrt{2} + \frac{3}{\sqrt{2}}$
46. $6\sqrt{3} + \frac{2}{\sqrt{3}}$
47. $\sqrt{8} - \frac{12}{\sqrt{2}} - 3\sqrt{2}$
48. $\sqrt{27} - \frac{6}{\sqrt{3}} - 5\sqrt{3}$

In Exercises 49-60, multiply to expand each of the given radical expressions. Place your final answer in simple radical form. Use your calculator to check your result.

49. $(2 + \sqrt{3})(3 - \sqrt{3})$
50. $(5 + \sqrt{2})(2 - \sqrt{2})$
51. $(4 + 3\sqrt{2})(2 - 5\sqrt{2})$
52. $(3 + 5\sqrt{3})(1 - 2\sqrt{3})$
53. $(2 + 3\sqrt{2})(2 - 3\sqrt{2})$
54. $(3 + 2\sqrt{3})(3 - 2\sqrt{3})$
55. $(2\sqrt{3} + 3\sqrt{2})(2\sqrt{3} - 3\sqrt{2})$
56. $(8\sqrt{2} + \sqrt{5})(8\sqrt{2} - \sqrt{5})$
57. $(2 + \sqrt{3})^2$
58. $(3 - \sqrt{2})^2$
59. $(\sqrt{3} - 2\sqrt{5})^2$
60. $(2\sqrt{3} + 3\sqrt{2})^2$

In Exercises 61-68, place each of the given rational expressions in simple radical form by “rationalizing the denominator.” Check your result with your calculator.

61. $\frac{1}{\sqrt{5} + \sqrt{3}}$
62. $\frac{1}{2\sqrt{3} - \sqrt{2}}$
63. $\frac{6}{2\sqrt{5} - \sqrt{2}}$
64. $\frac{9}{3\sqrt{3} - \sqrt{6}}$
65. \( \frac{2 + \sqrt{3}}{2 - \sqrt{3}} \)

66. \( \frac{3 - \sqrt{5}}{3 + \sqrt{5}} \)

67. \( \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}} \)

68. \( \frac{2\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}} \)

In Exercises 69-76, use the quadratic formula to find the solutions of the given equation. Place your solutions in simple radical form and reduce your solutions to lowest terms.

69. \( 3x^2 - 8x = 5 \)

70. \( 5x^2 - 2x = 1 \)

71. \( 5x^2 = 2x + 1 \)

72. \( 3x^2 - 2x = 11 \)

73. \( 7x^2 = 6x + 2 \)

74. \( 11x^2 + 6x = 4 \)

75. \( x^2 = 2x + 19 \)

76. \( 100x^2 = 40x - 1 \)

78. Given \( f(x) = \sqrt{x + 2} \), evaluate the expression

\[ \frac{f(x) - f(3)}{x - 3}, \]

and then “rationalize the numerator.”

79. Given \( f(x) = \sqrt{x} \), evaluate the expression

\[ \frac{f(x + h) - f(x)}{h}, \]

and then “rationalize the numerator.”

80. Given \( f(x) = \sqrt{x - 3} \), evaluate the expression

\[ \frac{f(x + h) - f(x)}{h}, \]

and then “rationalize the numerator.”
9.4 Answers

1. \(10\sqrt{7}\)
2. \(-2\sqrt{15}\)
3. \(15\sqrt{2}\)
4. \(-6\sqrt{15}\)
5. \(-24\sqrt{2}\)
6. 12
7. \(-50\)
8. \(6 + 2\sqrt{5}\)
9. \(-10 + 8\sqrt{2}\)
10. \(2\sqrt{2} + 2\)
11. \(2\sqrt{5} + 2\sqrt{7}\)
12. \(2\sqrt{2}\)
13. \(2\sqrt{2}\)
14. \(-6\sqrt{6}\)
15. \(5\sqrt{3} - 4\sqrt{2}\)
16. \(7\sqrt{2} - 5\sqrt{3}\)
17. \(5\sqrt{5}\)
18. \(4\sqrt{2}\)
19. \(-5\sqrt{3}\)
20. \(20\sqrt{5}\)
21. \(16\sqrt{5}\)
22. \(\sqrt{2}/2\)
23. \(\sqrt{2}/2\)
24. \(13\sqrt{2}/2\)
25. \(-7\sqrt{2}\)
26. \(3 + \sqrt{3}\)
27. \(-22 - 14\sqrt{2}\)
28. \(-14\)
29. \(-6\)
30. \(9 + 4\sqrt{5}\)
31. \(23 - 4\sqrt{15}\)
32. \(\sqrt{5} - \sqrt{3}/2\)
33. \(2\sqrt{5} + \sqrt{2}/3\)
34. \(7 + 4\sqrt{3}\)
35. \(5 + 2\sqrt{6}\)
36. \(9 + 4\sqrt{5}\)
37. \(23\sqrt{2}/3\)
38. \(1 + 2\sqrt{5}\)
39. \(1/(\sqrt{x} + \sqrt{2})\)
40. \(1/(\sqrt{x} + h + \sqrt{x})\)

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9.5 Radical Equations

In this section we are going to solve equations that contain one or more radical expressions. In the case where we can isolate the radical expression on one side of the equation, we can simply raise both sides of the equation to a power that will eliminate the radical expression. For example, if

\[ \sqrt{x - 1} = 2, \]  

then we can square both sides of the equation, eliminating the radical.

\[ (\sqrt{x - 1})^2 = (2)^2 \]
\[ x - 1 = 4 \]

Now that the radical is eliminated, we can appeal to well understood techniques to solve the equation that remains. In this case, we need only add 1 to both sides of the equation to obtain

\[ x = 5. \]

This solution is easily checked. Substitute \( x = 5 \) in the original equation (1).

\[ \sqrt{x - 1} = 2 \]
\[ \sqrt{5 - 1} = 2 \]
\[ \sqrt{4} = 2 \]

The last line is valid because the “positive square root of 4” is indeed 2.

This seems pretty straight forward, but there are some subtleties. Let’s look at another example, one with an equation quite similar to equation (1).

Example 2. Solve the equation \( \sqrt{x - 1} = -2 \) for \( x \).

If you carefully study the equation

\[ \sqrt{x - 1} = -2, \]  

you might immediately detect a difficulty. The left-hand side of the equation calls for a “positive square root,” but the right-hand side of the equation is negative. Intuitively, there can be no solutions.

A look at the graphs of each side of the equation also reveals the problem. The graphs of \( y = \sqrt{x - 1} \) and \( y = -2 \) are shown in Figure 1. Note that the graphs do not intersect, so the equation \( \sqrt{x - 1} = -2 \) has no solution.

However, note what happens when we square both sides of equation (3).

\[ (\sqrt{x - 1})^2 = (-2)^2 \]
\[ x - 1 = 4 \]  

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This result is identical to the result we got when we squared both sides of the equation \( \sqrt{x - 1} = 2 \) above. If we continue, adding 1 to both sides of the equation, we get

\[
x = 5.
\]

But this cannot be correct, as both intuition and the graphs in Figure 1 have shown that the equation \( \sqrt{x - 1} = -2 \) has no solutions.

Let’s check the solution \( x = 5 \) in the original equation (3).

\[
\begin{align*}
\sqrt{x - 1} &= -2 \\
\sqrt{5 - 1} &= -2 \\
\sqrt{4} &= -2
\end{align*}
\]

Because the “positive square root of 4” does not equal \(-2\), this last line is incorrect and the “solution” \( x = 5 \) does not check in the equation \( \sqrt{x - 1} = -2 \). Because the only solution we found does not check, the equation has no solutions.

The discussion in Example 2 dictates caution.

**Warning 5.** Whenever you square both sides of an equation, there is a possibility that you can introduce extraneous solutions, “extra” solutions that will not check in the original problem.

There is only one way to avoid this dilemma of extraneous equations.

**Checking Solutions.** Whenever you square both sides of an equation, you **must** check each of your solutions in the **original** equation. This is the only way you can be sure you have a valid solution.
Squaring a Binomial

As we’ve seen time and time again, the squaring a binomial pattern is of utmost importance.

**Squaring a Binomial.** If $a$ and $b$ are any real numbers, then

$$(a + b)^2 = a^2 + 2ab + b^2.$$  

The squaring a binomial pattern will play a major role in the rest of the examples in this section.

Let’s look at some examples of its use.

**Example 6.** Expand and simplify $(1 + \sqrt{x})^2$ by using the squaring a binomial pattern. Assume that $x \geq 0$.

The assumption that $x \geq 0$ is required, otherwise the expression $\sqrt{x}$ involves the square root of a negative number, which is not a real number.

The squaring a binomial pattern tells us to square the first and second terms. However, there is also a middle term, which is found by taking the product of the first and second terms, then multiplying the result by 2.

$$(1 + \sqrt{x})^2 = (1)^2 + 2(1)(\sqrt{x}) + (\sqrt{x})^2$$

$$= 1 + 2\sqrt{x} + x$$

Let’s look at another example.

**Example 7.** Expand and simplify $(\sqrt{x} + 1 - \sqrt{x})^2$ by using the squaring a binomial pattern. Comment on the domain of this expression.

In order for this expression to make sense, we must avoid taking the square root of a negative number. Hence, both expressions under the square roots must be nonnegative (positive or zero). That is,

$$x + 1 \geq 0 \quad \text{and} \quad x \geq 0$$

Solving each of these inequalities independently, we get the fact that

$$x \geq -1 \quad \text{and} \quad x \geq 0.$$  

Because of the word “and,” the requested domain is the set of all numbers that satisfy both inequalities, namely, the set of all real numbers that are greater than or equal to zero. That is, the domain of the expression is $\{x : x \geq 0\}$.

We will now expand the expression $(\sqrt{x} + 1 - \sqrt{x})^2$ using the squaring a binomial pattern.
\[(\sqrt{x+1} - \sqrt{x})^2 = (\sqrt{x+1})^2 - 2(\sqrt{x+1})(\sqrt{x}) + (\sqrt{x})^2\]
\[= x + 1 + 2\sqrt{(x+1)x + x}\]
\[= 2x + 1 + 2\sqrt{x^2 + x}\]

**Isolate the Radical**

Our mantra will be the strategy phrase “Isolate the radical.”

**Isolate the Radical.** When you solve equations containing one radical, isolate the radical by itself on one side of the equation.

Although this is not always possible (some equations might contain more than one radical expression), it is possible in our next example.

**Example 8.** Solve the equation

\[1 + \sqrt{4x + 13} = 2x\]  
(9)

for \(x\).

Let’s look at a graphing calculator solution. We’ve loaded the left- and right-hand sides of \(1 + \sqrt{4x + 13} = 2x\) into \(Y1\) and \(Y2\), respectively, as shown in Figure 2(a). We then use 6:ZStandard and the intersect utility on the CALC menu to determine the coordinates of the point of intersection of \(y = 1 + \sqrt{4x + 13}\) and \(y = 2x\), as shown in Figure 2(b).

![Graphing Calculator](graph.png)

(a) Loading \(y = 1 + \sqrt{4x + 13}\) and \(y = 2x\) into the \(Y=\) menu.

(b) The solution is \(x \approx 3\).

**Figure 2.** Solving \(1 + \sqrt{4x + 13} = 2x\) on the graphing calculator. Note that there is only one point of intersection.

We will now present an algebraic solution, but note that we are forewarned that there is only one solution and we believe that the solution is \(x \approx 3\). Of course, this is only an approximation, as is always the case when we pick up our calculator (our approximating machine).
Chant the strategy phrase “isolate the radical,” then isolate the radical on one side of the equation. We will accomplish this directive by subtracting 1 from both sides of the equation.

\[ 1 + \sqrt{4x + 13} = 2x \]
\[ \sqrt{4x + 13} = 2x - 1 \]

Next, square both sides of the equation.

\[ (\sqrt{4x + 13})^2 = (2x - 1)^2 \]

Squaring eliminates the radical on the left, but we must use the squaring a binomial pattern to square the binomial on the right-side of the equation.

\[ 4x + 13 = (2x)^2 - 2(2x)(1) + (1)^2 \]
\[ 4x + 13 = 4x^2 - 4x + 1 \]

We’ve succeed in clearing all square roots from the equation with our “isolate the radical” strategy. The equation that remains is nonlinear (there is a power of \(x\) higher than 1), so we want to make one side of the equation equal to zero. We will do this by subtracting \(4x\) and 13 from both sides of the equation.

\[ 0 = 4x^2 - 4x + 1 - 4x - 13 \]
\[ 0 = 4x^2 - 8x - 12 \]

At this point, note that each term on the right-hand side of the equation is divisible by 4. Divide both sides of the equation by 4, then use the \(ac\)-test to factor the result.

\[ 0 = x^2 - 2x - 3 \]
\[ 0 = (x - 3)(x + 1) \]

Set each factor on the right-hand side of this last equation to obtain the solutions \(x = 3\) and \(x = -1\).

Note that \(x = 3\) matches the solution found by graphing in Figure 2(b). However, an “extra” solution \(x = -1\) has appeared. Remember that we squared both sides of the original equation, so it is possible that extraneous solutions have been introduced. We need to check each of our solutions by substituting them into the original equation.

Our graph in Figure 2(b) adds credence to the analytical solution \(x = 3\), so let’s check that one first. Substitute \(x = 3\) in the original equation.

\[ 1 + \sqrt{4x + 13} = 2x \]
\[ 1 + \sqrt{4(3) + 13} = 2(3) \]
\[ 1 + \sqrt{25} = 6 \]
\[ 1 + 5 = 6 \]

Clearly, \(x = 3\) checks and is a valid solution.
Next, let’s check the “suspect” solution $x = -1$ by substituting it into the original equation.

\[
1 + \sqrt{4x + 13} = 2x \\
1 + \sqrt{4(-1) + 13} = 2(-1) \\
1 + \sqrt{9} = -2 \\
1 + 3 = -2
\]

Clearly, $x = -1$ does not check and is not a solution.

Thus, the only solution of $1 + \sqrt{4x + 13} = 2x$ is $x = 3$. Readers should take note how that graphical solution and the analytic solution complement one another.

Before looking at another example, let’s look at one of the most common mistakes made in the algebraic solution of equation (9).

A Common Algebraic Mistake

In this section we discuss one of the most common algebraic mistakes encountered when solving equations that contain radical expressions.

**Warning 10.** Many of the computations in this section are incorrect. They are examples of common algebra mistakes made when solving equations containing radicals. Keep this in mind and read the material in this section very carefully.

When presented with the equation

\[
1 + \sqrt{4x + 13} = 2x, \quad (11)
\]

some will square both sides of the equation in the following manner.

\[
(1)^2 + (\sqrt{4x + 13})^2 = (2x)^2, \quad (12)
\]

arriving at

\[
1 + 4x + 13 = 4x^2.
\]

Make one side zero, then divide both sides of the resulting equation by 2.

\[
0 = 4x^2 - 4x - 14 \\
0 = 2x^2 - 2x - 7
\]
The careful reader will already realize that we’ve traveled the wrong path, as this result is quite different from that at a similar point in the solution of Example 8. However, we can continue with the solution by using the quadratic formula to solve the last equation for $x$. When we compare $2x^2 - 2x - 7$ with $ax^2 + bx + c$, note that $a = 2$, $b = -2$, and $c = -7$. Thus,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-7)}}{2(2)}$$

$$= \frac{2 \pm \sqrt{60}}{4}.$$

However, neither of these “solutions” represent the correct solution found in Example 8, namely, $x = 3$. So, what have we done wrong?

The mistake occurred in the very first step when we squared both sides of the equation (11). Indeed, to get equation (12), we did not actually square both sides of equation (11). Rather, we squared each of the individual terms on each side of the equation.

This is a serious mistake. In essence, we started with an equation having the form

$$a + b = c,$$  \hspace{1cm} (13)

then squared “both sides” in the following manner.

$$a^2 + b^2 = c^2.$$  \hspace{1cm} (14)

This is not valid. For example, start with

$$2 + 3 = 5,$$

a completely valid equation as the sum of 2 and 3 is 5. Now “square” as we did in equation (14) to get

$$2^2 + 3^2 = 5^2.$$

However, note that this simplifies as

$$4 + 9 = 25,$$

so we no longer have a valid equation.

The mistake made here is that we squared each of the individual terms on each side of the equation instead of squaring “each side” of the equation. If we had done that, we would have been all right, as is seen in this calculation.

$$2 + 3 = 5$$

$$(2 + 3)^2 = 5^2$$

$$2^2 + 2(2)(3) + 3^2 = 5^2$$

$$4 + 12 + 9 = 25$$

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Just remember, \(a + b = c\) does not imply \(a^2 + b^2 = c^2\).

**Warning 15.** We will now return to correct computations.

**More than One Radical**

Let’s look at an equation that contains more than one radical.

**Example 16.** Solve the equation

\[
\sqrt{2x} + \sqrt{2x + 3} = 3
\]

(17)

for \(x\).

We’ll start with a graphical solution of the equation. First, load the equations \(y = \sqrt{2x} + \sqrt{2x + 3}\) and \(y = 3\) into the \(Y=\) menu, as shown in Figure 3(a).

We cannot take the square root of a negative number, so when we consider the function defined by the equation \(y = \sqrt{2x} + \sqrt{2x + 3}\), both expressions under the radicals must be nonnegative. That is,

\[
2x \geq 0 \quad \text{and} \quad 2x + 3 \geq 0.
\]

Solving each of these independently,

\[
x \geq 0 \quad \text{and} \quad x \geq -\frac{3}{2}.
\]

The numbers that are greater than or equal to zero and greater than or equal to \(-3/2\) are the numbers greater than or equal to zero. Hence, the domain of the function defined by the equation \(y = \sqrt{2x} + \sqrt{2x + 3}\) is \(\{x : x \geq 0\}\). Thus, it should not come as a shock when the graph of \(y = \sqrt{2x} + \sqrt{2x + 3}\) lies entirely to the right of zero, as shown in Figure 3(b).

![Figure 3. Drawing the graphs of \(y = \sqrt{2x} + \sqrt{2x + 3}\) and \(y = 3\).](image)
It’s a bit difficult to see the point of intersection in Figure 3(b), so let’s adjust the WINDOW settings as shown in Figure 4(a). As you can see Figure 4(b), this highlights the point of intersection a bit more clearly and the 5:intersect utility in the CALC menu finds the point of intersection shown in Figure 4(b).

(a) Adjust the view. (b) Use 5:intersect to find the point of intersection.

Figure 4. Solving $\sqrt{2x} + \sqrt{2x} + 3 = 3$ graphically.

The graphing calculator reports one solution (there’s only one point of intersection), and the $x$-value of the point of intersection is approximately $x \approx 0.5$.

Now, let’s look at an algebraic solution. Since are two radical expressions in this equation, we will isolate one of them on one side of the equation. We choose to isolate the more complex of the two radical expressions on the left-hand side of the equation, then square both sides of the resulting equation.

\[
\sqrt{2x} + \sqrt{2x} + 3 = 3
\]
\[
\sqrt{2x} + 3 = 3 - \sqrt{2x}
\]
\[
(\sqrt{2x} + 3)^2 = (3 - \sqrt{2x})^2
\]

On the left, squaring eliminates the radical. To square the binomial on the right, we use the squaring a binomial pattern to obtain

\[
2x + 3 = (3)^2 - 2(3)(\sqrt{2x}) + (\sqrt{2x})^2
\]
\[
2x + 3 = 9 - 6\sqrt{2x} + 2x.
\]

We still have one radical expression left on the right-hand side of this equation, so we’ll follow the mantra “isolate the radical.” First, subtract $2x$ from both sides of the equation to obtain

\[
3 = 9 - 6\sqrt{2x},
\]

then subtract 9 from both sides of the equation.

\[
-6 = -6\sqrt{2x}
\]
We’ve succeeded in isolating the radical term on one side of the equation. Now, divide both sides of the equation by $-6$, then square both sides of the resulting equation.

$$1 = \sqrt{2x}$$
$$1^2 = (\sqrt{2x})^2$$
$$1 = 2x$$

Divide both sides of the last result by 2.

$$x = \frac{1}{2}$$

Note that this agrees nicely with our graphical solution ($x \approx 0.5$), but let’s check our solution by substituting $x = 1/2$ into the original equation.

$$\sqrt{2x} + \sqrt{2x} + 3 = 3$$
$$\sqrt{2(1/2)} + \sqrt{2(1/2)} + 3 = 3$$
$$\sqrt{1} + \sqrt{4} = 3$$
$$1 + 2 = 3$$

This last statement is true, so the solution $x = 1/2$ checks.
9.5 Exercises

For the rational functions in Exercises 1-6, perform each of the following tasks.

i. Load the function \( f \) and the line \( y = k \) into your graphing calculator. Adjust the viewing window so that all point(s) of intersection of the two graphs are visible in your viewing window.

ii. Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label the graphs with their equations. Remember to draw all lines with a ruler.

iii. Use the \texttt{intersect} utility to determine the coordinates of the point(s) of intersection. Plot the point of intersection on your homework paper and label it with its coordinates.

iv. Solve the equation \( f(x) = k \) algebraically. Place your work and solution next to your graph. Do the solutions agree?

1. \( f(x) = \sqrt{x + 3}, \) \( k = 2 \)
2. \( f(x) = \sqrt{4 - x}, \) \( k = 3 \)
3. \( f(x) = \sqrt{7 - 2x}, \) \( k = 4 \)
4. \( f(x) = \sqrt{3}x + 5, \) \( k = 5 \)
5. \( f(x) = \sqrt{5} + x, \) \( k = 4 \)
6. \( f(x) = \sqrt{4 - x}, \) \( k = 5 \)

In Exercises 7-12, use an algebraic technique to solve the given equation. Check your solutions.

7. \( \sqrt{-5x + 5} = 2 \)
8. \( \sqrt{4x + 6} = 7 \)
9. \( \sqrt{6x - 8} = 8 \)
10. \( \sqrt{2x + 4} = 2 \)
11. \( \sqrt{-3x + 1} = 3 \)
12. \( \sqrt{4x + 7} = 3 \)

For the rational functions in Exercises 13-16, perform each of the following tasks.

i. Load the function \( f \) and the line \( y = k \) into your graphing calculator. Adjust the viewing window so that all point(s) of intersection of the two graphs are visible in your viewing window.

ii. Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label the graphs with their equations. Remember to draw all lines with a ruler.

iii. Use the \texttt{intersect} utility to determine the coordinates of the point(s) of intersection. Plot the point of intersection on your homework paper and label it with its coordinates.

iv. Solve the equation \( f(x) = k \) algebraically. Place your work and solution next to your graph. Do the solutions agree?

13. \( f(x) = \sqrt{x + 3} + x, \) \( k = 9 \)
14. \( f(x) = \sqrt{x + 6} - x, \) \( k = 4 \)
15. \( f(x) = \sqrt{x - 5} - x, \) \( k = -7 \)
16. \( f(x) = \sqrt{x + 5} + x, \) \( k = 7 \)
In Exercises 17-24, use an algebraic technique to solve the given equation. Check your solutions.

17. \(\sqrt{x + 1} + x = 5\)
18. \(\sqrt{x + 8} - x = 8\)
19. \(\sqrt{x + 4} + x = 8\)
20. \(\sqrt{x + 8} - x = 2\)
21. \(\sqrt{x + 5} - x = 3\)
22. \(\sqrt{x + 5} + x = 7\)
23. \(\sqrt{x + 9} - x = 9\)
24. \(\sqrt{x + 7} + x = 5\)

For the rational functions in Exercises 25-28, perform each of the following tasks.

i. Load the function \(f\) and the line \(y = k\) into your graphing calculator. Adjust the viewing window so that all point(s) of intersection of the two graphs are visible in your viewing window.
ii. Copy the image in your viewing window onto your homework paper. Label and scale each axis with \(xmin, xmax, ymin,\) and \(ymax\). Label the graphs with their equations. Remember to draw all lines with a ruler.
iii. Use the intersect utility to determine the coordinates of the point(s) of intersection. Plot the point of intersection on your homework paper and label it with its coordinates.
iv. Solve the equation \(f(x) = k\) algebraically. Place your work and solution next to your graph. Do the solutions agree?

25. \(f(x) = \sqrt{x - 1} + \sqrt{x + 6}, k = 7\)
26. \(f(x) = \sqrt{x + 2} + \sqrt{x + 9}, k = 7\)

In Exercises 29-40, use an algebraic technique to solve the given equation. Check your solutions.

27. \(f(x) = \sqrt{x + 2} + \sqrt{3x + 4}, k = 2\)
28. \(f(x) = \sqrt{6x + 7} + \sqrt{3x + 3}, k = 1\)

29. \(\sqrt{x + 6} - \sqrt{x - 35} = 1\)
30. \(\sqrt{x - 16} + \sqrt{x + 16} = 8\)
31. \(\sqrt{x - 19} + \sqrt{x - 6} = 13\)
32. \(\sqrt{x + 31} - \sqrt{x + 12} = 1\)
33. \(\sqrt{x - 2} - \sqrt{x - 49} = 1\)
34. \(\sqrt{x + 13} + \sqrt{x + 8} = 5\)
35. \(\sqrt{x + 27} - \sqrt{x - 22} = 1\)
36. \(\sqrt{x + 10} + \sqrt{x + 13} = 3\)
37. \(\sqrt{x + 30} - \sqrt{x - 38} = 2\)
38. \(\sqrt{x + 36} - \sqrt{x + 11} = 1\)
39. \(\sqrt{x - 17} + \sqrt{x + 3} = 10\)
40. \(\sqrt{x + 18} + \sqrt{x + 13} = 5\)
### 9.5 Answers

1. $x = 1$

3. $x = -9/2$

5. $x = 11$

7. $\frac{1}{5}$

9. 12

11. $\frac{8}{3}$

13. $x = 6$

15. $x = 9$

17. 3

19. 5

21. $-1$
23. $-8, -9$

25. $x = 10$

27. $x = -1$

29. 1635

31. 55

33. 578

35. 598

37. 294

39. 33
9.6 The Pythagorean Theorem

Pythagoras was a Greek mathematician and philosopher, born on the island of Samos (ca. 582 BC). He founded a number of schools, one in particular in a town in southern Italy called Crotone, whose members eventually became known as the Pythagoreans. The inner circle at the school, the *Mathematikoi*, lived at the school, rid themselves of all personal possessions, were vegetarians, and observed a strict vow of silence. They studied mathematics, philosophy, and music, and held the belief that numbers constitute the true nature of things, giving numbers a mystical or even spiritual quality.

Today, nothing is known of Pythagoras’s writings, perhaps due to the secrecy and silence of the Pythagorean society. However, one of the most famous theorems in all of mathematics does bear his name, the *Pythagorean Theorem*.

Let $c$ represent the length of the hypotenuse, the side of a right triangle directly opposite the right angle (a right angle measures $90^\circ$) of the triangle. The remaining sides of the right triangle are called the legs of the right triangle, whose lengths are designated by the letters $a$ and $b$.

The relationship involving the legs and hypotenuse of the right triangle, given by $a^2 + b^2 = c^2$, (1) is called the *Pythagorean Theorem*.

Note that the Pythagorean Theorem can only be applied to right triangles.

Let’s look at a simple application of the Pythagorean Theorem (1).

**Example 2.** Given that the length of one leg of a right triangle is 4 centimeters and the hypotenuse has length 8 centimeters, find the length of the second leg.

Let’s begin by sketching and labeling a right triangle with the given information. We will let $x$ represent the length of the missing leg.
Here is an important piece of advice.

**Tip 3.** The hypotenuse is the longest side of the right triangle. It is located directly opposite the right angle of the triangle. Most importantly, it is the quantity that is isolated by itself in the Pythagorean Theorem.

\[ a^2 + b^2 = c^2 \]

Always isolate the quantity representing the hypotenuse on one side of the equation. The legs go on the other side of the equation.

So, taking the tip to heart, and noting the lengths of the legs and hypotenuse in Figure 1, we write

\[ 4^2 + x^2 = 8^2. \]

Square, then isolate \( x \) on one side of the equation.

\[ 16 + x^2 = 64 \]
\[ x^2 = 48 \]

Normally, we would take plus or minus the square root in solving this equation, but \( x \) represents the length of a leg, which must be a positive number. Hence, we take just the positive square root of 48.

\[ x = \sqrt{48} \]

Of course, place your answer in simple radical form.

\[ x = \sqrt{16\sqrt{3}} \]
\[ x = 4\sqrt{3} \]

If need be, you can use your graphing calculator to approximate this length. To the nearest hundredth of a centimeter, \( x \approx 6.93 \) centimeters.
Proof of the Pythagorean Theorem

It is not known whether Pythagoras was the first to provide a proof of the Pythagorean Theorem. Many mathematical historians think not. Indeed, it is not even known if Pythagoras crafted a proof of the theorem that bears his name, let alone was the first to provide a proof.

There is evidence that the ancient Babylonians were aware of the Pythagorean Theorem over a 1000 years before the time of Pythagoras. A clay tablet, now referred to as Plimpton 322 (see Figure 2), contains examples of Pythagorean Triples, sets of three numbers that satisfy the Pythagorean Theorem (such as 3, 4, 5).

![Figure 2. A photograph of Plimpton 322.](image)

One of the earliest recorded proofs of the Pythagorean Theorem dates from the Han dynasty (206 BC to AD 220), and is recorded in the Chou Pei Suan Ching (see Figure 3). You can see that this figure specifically addresses the case of the 3, 4, 5 right triangle. Mathematical historians are divided as to whether or not the image was meant to be part of a general proof or was just devised to address this specific case. There is also disagreement over whether the proof was provided by a more modern commentator or dates back further in time.
However, Figure 3 does suggest a path we might take on the road to a proof of the Pythagorean Theorem. Start with an arbitrary right triangle having legs of lengths $a$ and $b$, and hypotenuse having length $c$, as shown in Figure 4(a).

Next, make four copies of the triangle shown in Figure 4(a), then rotate and translate them into place as shown in Figure 4(b). Note that this forms a big square that is $c$ units on a side.

Further, the position of the triangles in Figure 4(b) allows for the formation of a smaller, unshaded square in the middle of the larger square. It is not hard to calculate the length of the side of this smaller square. Simply subtract the length of the smaller leg from the larger leg of the original triangle. Thus, the side of the smaller square has length $b - a$.

Now, we will calculate the area of the large square in Figure 4(b) in two separate ways.

- First, the large square in Figure 4(b) has a side of length $c$. Therefore, the area of the large square is

  $$\text{Area} = c^2.$$ 

- Secondly, the large square in Figure 4(b) is made up of 4 triangles of the same size and one smaller square having a side of length $b - a$. We can calculate the area of the large square by summing the area of the 4 triangles and the smaller square.
The area of the smaller square is \((b - a)^2\).

The area of each triangle is \(ab/2\). Hence, the area of four triangles of equal size is four times this number; i.e., \(4(ab/2)\).

Thus, the area of the large square is

\[
\text{Area} = \text{Area of small square} + 4 \cdot \text{Area of triangle} = (b - a)^2 + 4 \left( \frac{ab}{2} \right).
\]

We calculated the area of the larger square twice. The first time we got \(c^2\); the second time we got \((b - a)^2 + 4(ab/2)\). Therefore, these two quantities must be equal.

\[
c^2 = (b - a)^2 + 4 \left( \frac{ab}{2} \right)
\]

Expand the binomial and simplify.

\[
c^2 = b^2 - 2ab + a^2 + 2ab
\]
\[
c^2 = b^2 + a^2
\]

That is,

\[
a^2 + b^2 = c^2,
\]

and the Pythagorean Theorem is proven.

**Applications of the Pythagorean Theorem**

In this section we will look at a few applications of the Pythagorean Theorem, one of the most applied theorems in all of mathematics. Just ask your local carpenter.

The ancient Egyptians would take a rope with 12 equally spaced knots like that shown in Figure 5, and use it to square corners of their buildings. The tool was instrumental in the construction of the pyramids.

The Pythagorean theorem is also useful in surveying, cartography, and navigation, to name a few possibilities.

Let’s look at a few examples of the Pythagorean Theorem in action.

**Example 4.** One leg of a right triangle is 7 meters longer than the other leg. The length of the hypotenuse is 13 meters. Find the lengths of all sides of the right triangle.

Let \(x\) represent the length of one leg of the right triangle. Because the second leg is 7 meters longer than the first leg, the length of the second leg can be represented by the expression \(x + 7\), as shown in Figure 6, where we’ve also labeled the length of the hypotenuse (13 meters).
Figure 6. The second leg is 7 meters longer than the first.

Remember to isolate the length of the hypotenuse on one side of the equation representing the Pythagorean Theorem. That is,

\[ x^2 + (x + 7)^2 = 13^2. \]

Note that the legs go on one side of the equation, the hypotenuse on the other. Square and simplify. Remember to use the squaring a binomial pattern.

\[
\begin{align*}
2x^2 + 14x + 49 &= 169 \\
2x^2 + 14x + 49 &= 169
\end{align*}
\]

This equation is nonlinear, so make one side zero by subtracting 169 from both sides of the equation.

\[
\begin{align*}
2x^2 + 14x - 120 &= 0 \\
2x^2 + 14x - 120 &= 0
\end{align*}
\]

Note that each term on the left-hand side of the equation is divisible by 2. Divide both sides of the equation by 2.

\[
x^2 + 7x - 60 = 0
\]

Let’s use the quadratic formula with \( a = 1 \), \( b = 7 \), and \( c = -60 \).

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-7 \pm \sqrt{7^2 - 4(1)(-60)}}{2(1)}
\]

Simplify.

\[
x = \frac{-7 \pm \sqrt{289}}{2}
\]

Note that 289 is a perfect square \((17^2 = 289)\). Thus,

\[
x = \frac{-7 \pm 17}{2}.
\]

Thus, we have two solutions,

\[
x = 5 \quad \text{or} \quad x = -12.
\]
Because length must be a positive number, we eliminate \(-12\) from consideration. Thus, the length of the first leg is \(x = 5\) meters. The length of the second leg is \(x + 7\), or 12 meters.

**Check.** Checking is an easy matter. The legs are 5 and 12 meters, respectively, and the hypotenuse is 13 meters. Note that the second leg is 7 meters longer than the first. Also,

\[
5^2 + 12^2 = 25 + 144 = 169,
\]

which is the square of 13.

The integral sides of the triangle in the previous example, 5, 12, and 13, are an example of a *Pythagorean Triple*.

**Pythagorean Triple.** A set of positive integers \(a, b,\) and \(c\), is called a Pythagorean Triple if they satisfy the Pythagorean Theorem; that is, if

\[
a^2 + b^2 = c^2.
\]

If the greatest common factor of \(a, b,\) and \(c\) is 1, then the triple \((a, b, c)\) is called a *primitive* Pythagorean Triple.

Thus, for example, the Pythagorean Triple \((5, 12, 13)\) is primitive.

Let’s look at another example.

**Example 5.** If \((a, b, c)\) is a Pythagorean Triple, show that any positive integral multiple is also a Pythagorean Triple.

Thus, if the positive integers \((a, b, c)\) is a Pythagorean Triple, we must show that \((ka, kb, kc)\), where \(k\) is a positive integer, is also a Pythagorean Triple.

However, we know that

\[
a^2 + b^2 = c^2.
\]

Multiply both sides of this equation by \(k^2\).

\[
k^2a^2 + k^2b^2 = k^2c^2
\]

This last result can be written

\[
(ka)^2 + (kb)^2 = (kc)^2.
\]

Hence, \((ka, kb, kc)\) is a Pythagorean Triple.

Hence, because \((3, 4, 5)\) is a Pythagorean Triple, you can double everything to get another triple \((6, 8, 10)\). Note that \(6^2 + 8^2 = 10^2\) is easily checked. Similarly, tripling gives another triple \((9, 12, 15)\), and so on.
In Example 5, we showed that \((5, 12, 13)\) was a triple, so we can take multiples to generate other Pythagorean Triples, such as \((10, 24, 26)\) or \((15, 36, 39)\), and so on.

Formulae for generating Pythagorean Triples have been known since antiquity.

**Example 6.** The following formula for generating Pythagorean Triples was published in Euclid’s (325–265 BC) *Elements*, one of the most successful textbooks in the history of mathematics. If \(m\) and \(n\) are positive integers with \(m > n\), show

\[
\begin{align*}
a &= m^2 - n^2, \\
b &= 2mn, \\
c &= m^2 + n^2,
\end{align*}
\]

(7)
generates Pythagorean Triples.

We need only show that the formulae for \(a\), \(b\), and \(c\) satisfy the Pythagorean Theorem. With that in mind, let’s first compute \(a^2 + b^2\).

\[
a^2 + b^2 = (m^2 - n^2)^2 + (2mn)^2 = m^4 - 2m^2n^2 + n^4 + 4m^2n^2 = m^4 + 2m^2n^2 + n^4
\]

On the other hand,

\[
c^2 = (m^2 + n^2)^2 = m^4 + 2m^2n^2 + n^4
\]

Hence, \(a^2 + b^2 = c^2\), and the expressions for \(a\), \(b\), and \(c\) form a Pythagorean Triple.

It is both interesting and fun to generate Pythagorean Triples with the formulae from Example 6. Choose \(m = 4\) and \(n = 2\), then

\[
\begin{align*}
a &= m^2 - n^2 = (4)^2 - (2)^2 = 12, \\
b &= 2mn = 2(4)(2) = 16, \\
c &= m^2 + n^2 = (4)^2 + (2)^2 = 20.
\end{align*}
\]

It is easy to check that the triple \((12, 16, 20)\) will satisfy \(12^2 + 16^2 = 20^2\). Indeed, note that this triple is a multiple of the basic \((3, 4, 5)\) triple, so it must also be a Pythagorean Triple.

It can also be shown that if \(m\) and \(n\) are relatively prime, and are not both odd or both even, then the formulae in Example 6 will generate a primitive Pythagorean Triple. For example, choose \(m = 5\) and \(n = 2\). Note that the greatest common divisor of \(m = 5\) and \(n = 2\) is one, so \(m\) and \(n\) are relatively prime. Moreover, \(m\) is odd while \(n\) is even. These values of \(m\) and \(n\) generate

\[
\begin{align*}
a &= m^2 - n^2 = (5)^2 - (2)^2 = 21, \\
b &= 2mn = 2(5)(2) = 20, \\
c &= m^2 + n^2 = (5)^2 + (2)^2 = 29.
\end{align*}
\]
Note that
\[
21^2 + 20^2 = 441 + 400 = 841 = 29^2.
\]

Hence, \((21, 20, 29)\) is a Pythagorean Triple. Moreover, the greatest common divisor of 21, 20, and 29 is one, so \((21, 20, 29)\) is primitive.

The practical applications of the Pythagorean Theorem are numerous.

**Example 8.** A painter leans a 20 foot ladder against the wall of a house. The base of the ladder is on level ground 5 feet from the wall of the house. How high up the wall of the house will the ladder reach?

Consider the triangle in Figure 7. The hypotenuse of the triangle represents the ladder and has length 20 feet. The base of the triangle represents the distance of the base of the ladder from the wall of the house and is 5 feet in length. The vertical leg of the triangle is the distance the ladder reaches up the wall and the quantity we wish to determine.

![Figure 7](image)

**Figure 7.** A ladder leans against the wall of a house.

Applying the Pythagorean Theorem,
\[
5^2 + h^2 = 20^2.
\]

Again, note that the square of the length of the hypotenuse is the quantity that is isolated on one side of the equation.

Next, square, then isolate the term containing \(h\) on one side of the equation by subtracting 25 from both sides of the resulting equation.
\[
25 + h^2 = 400
\]
\[
h^2 = 375
\]
We need only extract the positive square root.

\[ h = \sqrt{375} \]

We could place the solution in simple form, that is, \( h = 5\sqrt{15} \), but the nature of the problem warrants a decimal approximation. Using a calculator and rounding to the nearest tenth of a foot,

\[ h \approx 19.4. \]

Thus, the ladder reaches about 19.4 feet up the wall.

**The Distance Formula**

We often need to calculate the distance between two points \( P \) and \( Q \) in the plane. Indeed, this is such a frequently recurring need, we’d like to develop a formula that will quickly calculate the distance between the given points \( P \) and \( Q \). Such a formula is the goal of this last section.

Let \( P(x_1, y_1) \) and \( Q(x_2, y_2) \) be two arbitrary points in the plane, as shown in Figure 8(a) and let \( d \) represent the distance between the two points.

To find the distance \( d \), first draw the right triangle \( \triangle PQR \), with legs parallel to the axes, as shown in Figure 8(b). Next, we need to find the lengths of the legs of the right triangle \( \triangle PQR \).

- The distance between \( P \) and \( R \) is found by subtracting the \( x \)-coordinate of \( P \) from the \( x \)-coordinate of \( R \) and taking the absolute value of the result. That is, the distance between \( P \) and \( R \) is \( |x_2 - x_1| \).

- The distance between \( R \) and \( Q \) is found by subtracting the \( y \)-coordinate of \( R \) from the \( y \)-coordinate of \( Q \) and taking the absolute value of the result. That is, the distance between \( R \) and \( Q \) is \( |y_2 - y_1| \).
We can now use the Pythagorean Theorem to calculate $d$. Thus,

$$d^2 = (|x_2 - x_1|)^2 + (|y_2 - y_1|)^2.$$ 

However, for any real number $a$,

$$(|a|)^2 = |a| \cdot |a| = |a^2| = a^2,$$

because $a^2$ is nonnegative. Hence, $(|x_2 - x_1|)^2 = (x_2 - x_1)^2$ and $(|y_2 - y_1|)^2 = (y_2 - y_1)^2$ and we can write

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Taking the positive square root leads to the Distance Formula.

**The Distance Formula.** Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two arbitrary points in the plane. The distance $d$ between the points $P$ and $Q$ is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \quad (9)$$

The direction of subtraction is unimportant. Because you square the result of the subtraction, you get the same response regardless of the direction of subtraction (e.g. $(5 - 2)^2 = (2 - 5)^2$). Thus, it doesn’t matter which point you designate as the point $P$, nor does it matter which point you designate as the point $Q$. Simply subtract $x$-coordinates and square, subtract $y$-coordinates and square, add, then take the square root.

Let’s look at an example.

**Example 10.** Find the distance between the points $P(-4, -2)$ and $Q(4, 4)$.

It helps the intuition if we draw a picture, as we have in Figure 9. One can now take a compass and open it to the distance between points $P$ and $Q$. Then you can place your compass on the horizontal axis (or any horizontal gridline) to estimate the distance between the points $P$ and $Q$. We did that on our graph paper and estimate the distance $d \approx 10$.

![Figure 9](image-url)
Let’s now use the distance formula to obtain an exact value for the distance $d$. With $(x_1, y_1) = P(-4, -2)$ and $(x_2, y_2) = Q(4, 4)$,

$$
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
= \sqrt{(4 - (-4))^2 + (4 - (-2))^2}
= \sqrt{8^2 + 6^2}
= \sqrt{64 + 36}
= \sqrt{100}
= 10.
$$

It’s not often that your exact result agrees with your approximation, so never worry if you’re off by just a little bit.
9.6 Exercises

In Exercises 1-8, state whether or not the given triple is a Pythagorean Triple. Give a reason for your answer.

1. (8, 15, 17)
2. (7, 24, 25)
3. (8, 9, 17)
4. (4, 9, 13)
5. (12, 35, 37)
6. (12, 17, 29)
7. (11, 17, 28)
8. (11, 60, 61)

In Exercises 9-16, set up an equation to model the problem constraints and solve. Use your answer to find the missing side of the given right triangle. Include a sketch with your solution and check your result.

9. 10. 

11. 12. 

13.
14.  

\[ 4\sqrt{3} \]

15.  

\[ 5 \quad 10 \]

16.  

\[ 8 \quad 8\sqrt{2} \]

17.  The legs of a right triangle are consecutive positive integers. The hypotenuse has length 5. What are the lengths of the legs?

18.  The legs of a right triangle are consecutive even integers. The hypotenuse has length 10. What are the lengths of the legs?

19.  One leg of a right triangle is 1 centimeter less than twice the length of the first leg. If the length of the hypotenuse is 17 centimeters, find the lengths of the legs.

20.  One leg of a right triangle is 3 feet longer than 3 times the length of the first leg. The length of the hypotenuse is 25 feet. Find the lengths of the legs.

21.  Pythagoras is credited with the following formulae that can be used to generate Pythagorean Triples.

\[
\begin{align*}
    a &= m \\
    b &= \frac{m^2 - 1}{2} \\
    c &= \frac{m^2 + 1}{2}
\end{align*}
\]

Use the technique of Example 6 to demonstrate that the formulae given above will generate Pythagorean Triples, provided that \( m \) is an odd positive integer larger than one. Secondly, generate at least 3 instances of Pythagorean Triples with Pythagoras’s formula.

22.  Plato (380 BC) is credited with the following formulae that can be used to generate Pythagorean Triples.

\[
\begin{align*}
    a &= 2m \\
    b &= m^2 - 1 \\
    c &= m^2 + 1
\end{align*}
\]

Use the technique of Example 6 to demonstrate that the formulae given above will generate Pythagorean Triples, provided that \( m \) is a positive integer larger than 1. Secondly, generate at least 3 instances of Pythagorean Triples with Plato’s formula.
In Exercises 23-28, set up an equation that models the problem constraints. Solve the equation and use the result to answer the question. Look back and check your result.

23. Fritz and Greta are planting a 12-foot by 18-foot rectangular garden, and are laying it out using string. They would like to know the length of a diagonal to make sure that right angles are formed. Find the length of a diagonal. Approximate your answer to within 0.1 feet.

24. Angelina and Markos are planting a 20-foot by 28-foot rectangular garden, and are laying it out using string. They would like to know the length of a diagonal to make sure that right angles are formed. Find the length of a diagonal. Approximate your answer to within 0.1 feet.

25. The base of a 36-foot long guy wire is located 16 feet from the base of the telephone pole that it is anchoring. How high up the pole does the guy wire reach? Approximate your answer to within 0.1 feet.

26. The base of a 35-foot long guy wire is located 10 feet from the base of the telephone pole that it is anchoring. How high up the pole does the guy wire reach? Approximate your answer to within 0.1 feet.

27. A stereo receiver is in a corner of a 13-foot by 16-foot rectangular room. Speaker wire will run under a rug, diagonally, to a speaker in the far corner. If 3 feet of slack is required on each end, how long a piece of wire should be purchased? Approximate your answer to within 0.1 feet.

28. A stereo receiver is in a corner of a 10-foot by 15-foot rectangular room. Speaker wire will run under a rug, diagonally, to a speaker in the far corner. If 4 feet of slack is required on each end, how long a piece of wire should be purchased? Approximate your answer to within 0.1 feet.

In Exercises 29-38, use the distance formula to find the exact distance between the given points.

29. \((-8, -9)\) and \((6, -6)\)
30. \((1, 0)\) and \((-9, -2)\)
31. \((-9, 1)\) and \((-8, 7)\)
32. \((0, 9)\) and \((3, 1)\)
33. \((6, -5)\) and \((-9, -2)\)
34. \((-9, 6)\) and \((1, 4)\)
35. \((-7, 7)\) and \((-3, 6)\)
36. \((-7, -6)\) and \((-2, -4)\)
37. \((4, -3)\) and \((-9, 6)\)
38. \((-7, -1)\) and \((4, -5)\)

In Exercises 39-42, set up an equation that models the problem constraints. Solve the equation and use the result to answer the question. Look back and check your result.

39. Find \(k\) so that the point \((4, k)\) is \(2\sqrt{2}\) units away from the point \((2, 1)\).
40. Find \(k\) so that the point \((k, 1)\) is \(2\sqrt{2}\) units away from the point \((0, -1)\).
41. Find $k$ so that the point $(k, 1)$ is $\sqrt{17}$ units away from the point $(2, -3)$.

42. Find $k$ so that the point $(-1, k)$ is $\sqrt{13}$ units away from the point $(-4, -3)$.

43. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Plot the points $P(0, 5)$ and $Q(4, -3)$ on your coordinate system.

a) Plot several points that are equidistant from the points $P$ and $Q$ on your coordinate system. What graph do you get if you plot all points that are equidistant from the points $P$ and $Q$? Determine the equation of the graph by examining the resulting image on your coordinate system.

b) Use the distance formula to find the equation of the graph of all points that are equidistant from the points $P$ and $Q$. Hint: Let $(x, y)$ represent an arbitrary point on the graph of all points equidistant from points $P$ and $Q$. Calculate the distances from the point $(x, y)$ to the points $P$ and $Q$ separately, then set them equal and simplify the resulting equation. Note that this analytical approach should provide an equation that matches that found by the graphical approach in part (a).

44. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Plot the point $P(0, 2)$ and label it with its coordinates. Draw the line $y = -2$ and label it with its equation.

a) Plot several points that are equidistant from the point $P$ and the line $y = -2$ on your coordinate system. What graph do you get if you plot all points that are equidistant from the points $P$ and the line $y = -2$.

b) Use the distance formula to find the equation of the graph of all points that are equidistant from the points $P$ and the line $y = -2$. Hint: Let $(x, y)$ represent an arbitrary point on the graph of all points equidistant from points $P$ and the line $y = -2$. Calculate the distances from the point $(x, y)$ to the points $P$ and the line $y = -2$ separately, then set them equal and simplify the resulting equation.
45. Copy the following figure onto a sheet of graph paper. Cut the pieces of the first figure out with a pair of scissors, then rearrange them to form the second figure. Explain how this proves the Pythagorean Theorem.

46. Compare this image to the one that follows and explain how this proves the Pythagorean Theorem.
9.6 Answers

1. Yes, because $8^2 + 15^2 = 17^2$

3. No, because $8^2 + 9^2 \neq 17^2$

5. Yes, because $12^2 + 35^2 = 37^2$

7. No, because $11^2 + 17^2 \neq 28^2$

9. 4

11. $4\sqrt{3}$

13. $2\sqrt{2}$

15. $5\sqrt{3}$

17. The legs have lengths 3 and 4.

19. The legs have lengths 8 and 15 centimeters.

21. $(3, 4, 5)$, $(5, 12, 13)$, and $(7, 24, 25)$, with $m = 3$, 5, and 7, respectively.

23. 21.63 ft

25. 32.25 ft

27. 26.62 ft

29. $\sqrt{205}$

31. $\sqrt{37}$

33. $\sqrt{234} = 3\sqrt{26}$

35. $\sqrt{17}$

37. $\sqrt{250} = 5\sqrt{10}$

39. $k = 3, -1$.

41. $k = 1, 3$.

43.

a) In the figure that follows, $XP = XQ$.

b) $y = (1/2)x$
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