9.1 Exercises

In Exercises 1–10, complete each of the following tasks.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis.

ii. Complete the table of points for the given function. Plot each of the points on your coordinate system, then use them to help draw the graph of the given function.

iii. Use different colored pencils to project all points onto the $x$- and $y$-axes to determine the domain and range. Use interval notation to describe the domain of the given function.

1. \( f(x) = -\sqrt{x} \)

2. \( f(x) = \sqrt{-x} \)

3. \( f(x) = \sqrt{x} + 2 \)

4. \( f(x) = \sqrt{5 - x} \)

5. \( f(x) = \sqrt{x} + 2 \)

6. \( f(x) = \sqrt{x} - 1 \)

7. \( f(x) = \sqrt{x + 3} + 2 \)

8. \( f(x) = \sqrt{x - 1} + 3 \)

9. \( f(x) = \sqrt{3 - x} \)

10. \( f(x) = -\sqrt{x + 3} \)

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In Exercises 11-20, perform each of the following tasks.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Remember to draw all lines with a ruler.

ii. Use geometric transformations to draw the graph of the given function on your coordinate system without the use of a graphing calculator. Note: You may check your solution with your calculator, but you should be able to produce the graph without the use of your calculator.

iii. Use different colored pencils to project the points on the graph of the function onto the x- and y-axes. Use interval notation to describe the domain and range of the function.

11. \( f(x) = \sqrt{x} + 3 \)

12. \( f(x) = \sqrt{x} + 3 \)

13. \( f(x) = \sqrt{x} - 2 \)

14. \( f(x) = \sqrt{x} - 2 \)

15. \( f(x) = \sqrt{x} + 5 + 1 \)

16. \( f(x) = \sqrt{x} - 2 - 1 \)

17. \( f(x) = -\sqrt{x} + 4 \)

18. \( f(x) = -\sqrt{x} + 4 \)

19. \( f(x) = -\sqrt{x} + 3 \)

20. \( f(x) = -\sqrt{x} + 3 \)

21. To draw the graph of the function \( f(x) = \sqrt{3 - x} \), perform each of the following steps in sequence without the aid of a calculator.

i. Set up a coordinate system and sketch the graph of \( y = \sqrt{x} \). Label the graph with its equation.

ii. Set up a second coordinate system and sketch the graph of \( y = \sqrt{-x} \). Label the graph with its equation.

iii. Set up a third coordinate system and sketch the graph of \( y = \sqrt{-x - 3} \). Label the graph with its equation. This is the graph of \( f(x) = \sqrt{3 - x} \). Use interval notation to state the domain and range of this function.

22. To draw the graph of the function \( f(x) = \sqrt{-x - 3} \), perform each of the following steps in sequence.

i. Set up a coordinate system and sketch the graph of \( y = \sqrt{x} \). Label the graph with its equation.

ii. Set up a second coordinate system and sketch the graph of \( y = \sqrt{-x} \). Label the graph with its equation.

iii. Set up a third coordinate system and sketch the graph of \( y = \sqrt{-x - 3} \). Label the graph with its equation. This is the graph of \( f(x) = \sqrt{3 - x} \). Use interval notation to state the domain and range of this function.

23. To draw the graph of the function \( f(x) = \sqrt{-x - 1} \), perform each of the following steps in sequence without the aid of a calculator.

i. Set up a coordinate system and sketch the graph of \( y = \sqrt{x} \). Label the graph with its equation.

ii. Set up a second coordinate system and sketch the graph of \( y = \sqrt{-x} \). Label the graph with its equation.

iii. Set up a third coordinate system and sketch the graph of \( y = \sqrt{-x - 1} \). Label the graph with its equation. This is the graph of \( f(x) = \sqrt{3 - x} \). Use interval notation to state the domain and range of this function.
24. To draw the graph of the function \( f(x) = \sqrt{1-x} \), perform each of the following steps in sequence.

i. Set up a coordinate system and sketch the graph of \( y = \sqrt{x} \). Label the graph with its equation.

ii. Set up a second coordinate system and sketch the graph of \( y = \sqrt{-x} \). Label the graph with its equation.

iii. Set up a third coordinate system and sketch the graph of \( y = \sqrt{-(x-1)} \). Label the graph with its equation. This is the graph of \( f(x) = \sqrt{1-x} \). Use interval notation to state the domain and range of this function.

In Exercises 29-40, find the domain of the given function algebraically.

29. \( f(x) = \sqrt{2x + 9} \)

30. \( f(x) = \sqrt{-3x + 3} \)

31. \( f(x) = \sqrt{-8x - 3} \)

32. \( f(x) = \sqrt{-3x + 6} \)

33. \( f(x) = \sqrt{-6x - 3} \)

34. \( f(x) = \sqrt{8x - 6} \)

35. \( f(x) = \sqrt{-7x + 2} \)

36. \( f(x) = \sqrt{8x - 3} \)

37. \( f(x) = \sqrt{6x + 3} \)

38. \( f(x) = \sqrt{x - 5} \)

39. \( f(x) = \sqrt{-7x - 8} \)

40. \( f(x) = \sqrt{7x + 8} \)

In Exercises 25-28, perform each of the following tasks.

i. Draw the graph of the given function with your graphing calculator. Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label your graph with its equation. Use the graph to determine the domain of the function and describe the domain with interval notation.

ii. Use a purely algebraic approach to determine the domain of the given function. Use interval notation to describe your result. Does it agree with the graphical result from part (i)?

25. \( f(x) = \sqrt{2x + 7} \)

26. \( f(x) = \sqrt{7 - 2x} \)

27. \( f(x) = \sqrt{12 - 4x} \)

28. \( f(x) = \sqrt{12 + 2x} \)
1. Complete the table for \( f(x) = -\sqrt{x} \).

\[
\begin{array}{|c|c|c|c|}
\hline
x & 0 & 1 & 4 \quad 9 \\
\hline
f(x) & 0 & -1 & -2 \quad -3 \\
\hline
\end{array}
\]

Plot the points in the table and use them to help draw the graph.

Project all points on the graph onto the \( x \)-axis to determine the domain: Domain = \([0, \infty)\). Project all points on the graph onto the \( y \)-axis to determine the range: Range = \(( -\infty, 0] \).

3. Complete the table for \( f(x) = \sqrt{x + 2} \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
x & -2 & -1 & 2 & 7 \\
\hline
f(x) & 0 & 1 & 2 & 3 \\
\hline
\end{array}
\]

Plot the points in the table and use them to help draw the graph.
Project all points on the graph onto the $x$-axis to determine the domain: Domain = $[-2, \infty)$. Project all points on the graph onto the $y$-axis to determine the range: Range = $[0, \infty)$.

5. Complete the table for $f(x) = \sqrt{x} + 2$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>4</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

Plot the points in the table and use them to draw the graph of $f$.

7. Complete the table for $f(x) = \sqrt{x+3} + 2$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>−3</th>
<th>−2</th>
<th>1</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

Plot the points in the table and use them to draw the graph of $f$. 

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Project all points on the graph onto the $x$-axis to determine the domain: Domain = $(-3, \infty)$. Project all points on the graph onto the $y$-axis to determine the range: Range = $[2, \infty)$.

9. Complete the table for $f(x) = \sqrt{3 - x}$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-6</th>
<th>-1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Plot the points in the table and use them to draw the graph of $f$.

Project all points on the graph onto the $x$-axis to determine the domain: Domain = $(-\infty, 3]$. Project all points on the graph onto the $y$-axis to determine the range: Range = $[0, \infty)$.

11. First, plot the graph of $y = \sqrt{x}$, as shown in (a). Then, add 3 to produce the equation $y = \sqrt{x} + 3$. This will shift the graph of of $y = \sqrt{x}$ upward 3 units, as shown in (b).

Project all points on the graph onto the $x$-axis to determine the domain: Domain = $[0, \infty)$. Project all points on the graph onto the $y$-axis to determine the range: Range = $[3, \infty)$.

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13. First, plot the graph of $y = \sqrt{x}$, as shown in (a). Then, replace $x$ with $x - 2$ to produce the equation $y = \sqrt{x - 2}$. This will shift the graph of $y = \sqrt{x}$ to the right 2 units, as shown in (b).

Project all points on the graph onto the $x$-axis to determine the domain: Domain = $[2, \infty)$. Project all points on the graph onto the $y$-axis to determine the range: Range = $[0, \infty)$.

15. First, plot the graph of $y = \sqrt{x}$, as shown in (a). Then, replace $x$ with $x + 5$ to produce the equation $y = \sqrt{x + 5}$. Then add 1 to produce the equation $y = \sqrt{x + 5} + 1$. 
This will shift the graph of of $y = \sqrt{x}$ to the left 5 units, then upward 1 unit, as shown in (b).

![Graphs showing transformations](image)

Project all points on the graph onto the $x$-axis to determine the domain: Domain = $[-5, \infty)$. Project all points on the graph onto the $y$-axis to determine the range: Range = $[1, \infty)$.

17. First, plot the graph of $y = \sqrt{x}$, as shown in (a). Then, negate to produce the equation $y = -\sqrt{x}$. This will reflect the graph of $y = \sqrt{x}$ across the $x$-axis as shown in (b). Finally, replace $x$ with $x + 4$ to produce the equation $y = -\sqrt{x + 4}$. This will shift the graph of $y = -\sqrt{x}$ four units to the left, as shown in (c).

![Graphs showing transformations](image)
Project all points on the graph onto the x-axis to determine the domain: Domain = \([-4, \infty)\). Project all points on the graph onto the y-axis to determine the range: Range = \((-\infty, 0]\).

19. First, plot the graph of \(y = \sqrt{x}\), as shown in (a). Then, negate to produce the equation \(y = -\sqrt{x}\). This will reflect the graph of \(y = \sqrt{x}\) across the x-axis as shown in (b). Finally, add 3 to produce the equation \(y = -\sqrt{x} + 3\). This will shift the graph of \(y = -\sqrt{x}\) three units upward, as shown in (c).

Project all points on the graph onto the x-axis to determine the domain: Domain = \([0, \infty)\). Project all points on the graph onto the y-axis to determine the range: Range = \((-\infty, 3]\).
21. First, plot the graph of $y = \sqrt{x}$, as shown in (a). Then, replace $x$ with $-x$ to produce the equation $y = \sqrt{-x}$. This will reflect the graph of $y = \sqrt{x}$ across the $y$-axis, as shown in (b). Finally, replace $x$ with $x - 3$ to produce the equation $y = \sqrt{-(x - 3)}$. This will shift the graph of $y = \sqrt{-x}$ three units to the right, as shown in (c).

Project all points on the graph onto the $x$-axis to determine the domain: Domain = $(-\infty, 3]$. Project all points on the graph onto the $y$-axis to determine the range: Range = $[0, \infty)$.

23. First, plot the graph of $y = \sqrt{x}$, as shown in (a). Then, replace $x$ with $-x$ to produce the equation $y = \sqrt{-x}$. This will reflect the graph of $y = \sqrt{x}$ across the $y$-axis, as shown in (b). Finally, replace $x$ with $x + 1$ to produce the equation $y = \sqrt{-x}$ one unit to the left, as shown in (c).
Section 9.1 The Square Root Function

Project all points on the graph onto the $x$-axis to determine the domain: Domain = $(-\infty, -1]$. Project all points on the graph onto the $y$-axis to determine the range: Range = $[0, \infty)$.

25. We use a graphing calculator to produce the following graph of $f(x) = \sqrt{2x + 7}$.

We estimate that the domain will consist of all real numbers to the right of approximately $-3.5$. To find an algebraic solution, note that you cannot take the square root of a negative number. Hence, the expression under the radical in $f(x) = \sqrt{2x + 7}$ must be greater than or equal to zero.
Chapter 9  Radical Functions

\[ 2x + 7 \geq 0 \]
\[ 2x \geq -7 \]
\[ x \geq -\frac{7}{2} \]

Hence, the domain is \([-\frac{7}{2}, \infty)\).

27. We use a graphing calculator to produce the following graph of \(f(x) = \sqrt{12 - 4x}\).

We estimate that the domain will consist of all real numbers to the left of approximately 3. To find an algebraic solution, note that you cannot take the square root of a negative number. Hence, the expression under the radical in \(f(x) = \sqrt{12 - 4x}\) must be greater than or equal to zero.

\[ 12 - 4x \geq 0 \]
\[ -4x \geq -12 \]
\[ x \leq 3 \]

Hence, the domain is \((-\infty, 3]\).

29. The even root of a negative number is not defined as a real number. Thus, \(2x + 9\) must be greater than or equal to zero. Since \(2x + 9 \geq 0\) implies that \(x \geq -\frac{9}{2}\), the domain is the interval \([-\frac{9}{2}, \infty)\).

31. The even root of a negative number is not defined as a real number. Thus, \(-8x - 3\) must be greater than or equal to zero. Since \(-8x - 3 \geq 0\) implies that \(x \leq -\frac{3}{8}\), the domain is the interval \((-\infty, -\frac{3}{8}]\).

33. The even root of a negative number is not defined as a real number. Thus, \(-6x - 8\) must be greater than or equal to zero. Since \(-6x - 8 \geq 0\) implies that \(x \leq -\frac{4}{3}\), the domain is the interval \((-\infty, -\frac{4}{3}]\).

35. The even root of a negative number is not defined as a real number. Thus, \(-7x + 2\) must be greater than or equal to zero. Since \(-7x + 2 \geq 0\) implies that \(x \leq \frac{2}{7}\), the domain is the interval \((-\infty, \frac{2}{7}]\).

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37. The even root of a negative number is not defined as a real number. Thus, $6x + 3$ must be greater than or equal to zero. Since $6x + 3 \geq 0$ implies that $x \geq -\frac{1}{2}$, the domain is the interval $\left[-\frac{1}{2}, \infty\right)$.

39. The even root of a negative number is not defined as a real number. Thus, $-7x - 8$ must be greater than or equal to zero. Since $-7x - 8 \geq 0$ implies that $x \leq -\frac{8}{7}$, the domain is the interval $(-\infty, -\frac{8}{7}]$. 
## 9.2 Exercises

1. Use a calculator to first approximate \( \sqrt{5} \sqrt{2} \). On the same screen, approximate \( \sqrt{10} \). Report the results on your homework paper.

2. Use a calculator to first approximate \( \sqrt{7} \sqrt{10} \). On the same screen, approximate \( \sqrt{70} \). Report the results on your homework paper.

3. Use a calculator to first approximate \( \sqrt{3} \sqrt{11} \). On the same screen, approximate \( \sqrt{33} \). Report the results on your homework paper.

4. Use a calculator to first approximate \( \sqrt{5} \sqrt{3} \). On the same screen, approximate \( \sqrt{65} \). Report the results on your homework paper.

In Exercises 5-20, place each of the radical expressions in simple radical form. As in Example 3 in the narrative, check your result with your calculator.

5. \( \sqrt{18} \)

6. \( \sqrt{80} \)

7. \( \sqrt{112} \)

8. \( \sqrt{72} \)

9. \( \sqrt{108} \)

10. \( \sqrt{54} \)

11. \( \sqrt{50} \)

12. \( \sqrt{48} \)

13. \( \sqrt{245} \)

14. \( \sqrt{150} \)

15. \( \sqrt{98} \)

16. \( \sqrt{252} \)

17. \( \sqrt{45} \)

18. \( \sqrt{294} \)

19. \( \sqrt{24} \)

20. \( \sqrt{32} \)

In Exercises 21-26, use prime factorization (as in Examples 10 and 11 in the narrative) to assist you in placing the given radical expression in simple radical form. Check your result with your calculator.

21. \( \sqrt{2016} \)

22. \( \sqrt{2700} \)

23. \( \sqrt{14175} \)

24. \( \sqrt{44000} \)

25. \( \sqrt{20250} \)

26. \( \sqrt{3564} \)

In Exercises 27-46, place each of the given radical expressions in simple radical form. Make no assumptions about the sign of the variables. Variables can either represent positive or negative numbers.

27. \( \sqrt{(6x - 11)^4} \)

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28. \(\sqrt{16m^8}\)

29. \(\sqrt{25f^2}\)

30. \(\sqrt{25f^8}\)

31. \(\sqrt{16m^2}\)

32. \(\sqrt{25a^2}\)

33. \((7x + 5)^{12}\)

34. \(\sqrt{9w^{10}}\)

35. \(\sqrt{25x^2 - 50x + 25}\)

36. \(\sqrt{49x^2 - 42x + 9}\)

37. \(\sqrt{25x^2 + 90x + 81}\)

38. \(\sqrt{25f^{14}}\)

39. \((3x + 6)^{12}\)

40. \((9x - 8)^{12}\)

41. \((36x^2 + 36x + 9)^{12}\)

42. \(\sqrt{4e^2}\)

43. \(\sqrt{4p^{10}}\)

44. \(\sqrt{25x^{12}}\)

45. \(\sqrt{25q^6}\)

46. \(\sqrt{16h^{12}}\)

47. Given that \(x < 0\), place the radical expression \(\sqrt{32x^6}\) in simple radical form. Check your solution on your calculator for \(x = -2\).

48. Given that \(x < 0\), place the radical expression \(\sqrt{54x^9}\) in simple radical form. Check your solution on your calculator for \(x = -2\).

49. Given that \(x < 0\), place the radical expression \(\sqrt{27x^{12}}\) in simple radical form. Check your solution on your calculator for \(x = -2\).

50. Given that \(x < 0\), place the radical expression \(\sqrt{44x^{10}}\) in simple radical form. Check your solution on your calculator for \(x = -2\).

51. Given that \(x < 4\), place the radical expression \(\sqrt{x^2 - 8x + 16}\) in simple radical form. Use a graphing calculator to show that the graphs of the original expression and your simple radical form agree for all values of \(x\) such that \(x < 4\).

52. Given that \(x \geq -2\), place the radical expression \(\sqrt{x^2 + 4x + 4}\) in simple radical form. Use a graphing calculator to show that the graphs of the original expression and your simple radical form agree for all values of \(x\) such that \(x \geq -2\).

53. Given that \(x \geq 5\), place the radical expression \(\sqrt{x^2 - 10x + 25}\) in simple radical form. Use a graphing calculator to show that the graphs of the original expression and your simple radical form agree for all values of \(x\) such that \(x \geq 5\).

54. Given that \(x < -1\), place the radical expression \(\sqrt{x^2 + 2x + 1}\) in simple radical form. Use a graphing calculator to show that the graphs of the original expression and your simple radical form agree for all values of \(x\) such that \(x < -1\).
In Exercises 55-72, place each radical expression in simple radical form. Assume that all variables represent positive numbers.

55. $\sqrt{9e^{13}}$
56. $\sqrt{4k^{2}}$
57. $\sqrt{25x^{2} + 40x + 16}$
58. $\sqrt{9x^{2} - 30x + 25}$
59. $\sqrt{4j^{11}}$
60. $\sqrt{16j^{10}}$
61. $\sqrt{25m^{2}}$
62. $\sqrt{9e^{9}}$
63. $\sqrt{4e^{5}}$
64. $\sqrt{25z^{2}}$
65. $\sqrt{25h^{10}}$
66. $\sqrt{25b^{2}}$
67. $\sqrt{9s^{7}}$
68. $\sqrt{9e^{4}}$
69. $\sqrt{4p^{8}}$
70. $\sqrt{9d^{15}}$
71. $\sqrt{9q^{10}}$
72. $\sqrt{4w^{7}}$

In Exercises 73-80, place each given radical expression in simple radical form. Assume that all variables represent positive numbers.

73. $\sqrt{2f^{5}} \sqrt{8f^{3}}$
74. $\sqrt{3s^{3}} \sqrt{243s^{3}}$
75. $\sqrt{2k^{7}} \sqrt{32k^{3}}$
76. $\sqrt{2n^{9}} \sqrt{8n^{3}}$
77. $\sqrt{2e^{9}} \sqrt{8e^{5}}$
78. $\sqrt{5n^{9}} \sqrt{125n^{3}}$
79. $\sqrt{3z^{5}} \sqrt{27z^{3}}$
80. $\sqrt{3t^{7}} \sqrt{27t^{3}}
9.2 Solutions

1. Note that $\sqrt{5\sqrt{2}} = \sqrt{10} \approx 3.16227766$.

3. Note that $\sqrt{3\sqrt{11}} = \sqrt{33} \approx 5.744562647$.

5.

$\sqrt{18} = \sqrt{3^2 \cdot 2} = \sqrt{3^2} \sqrt{2} = 3\sqrt{2}$

7.

$\sqrt{112} = \sqrt{4^2 \cdot 7} = \sqrt{4^2} \sqrt{7} = 4\sqrt{7}$

9.

$\sqrt{108} = \sqrt{6^2 \cdot 3} = \sqrt{6^2} \sqrt{3} = 6\sqrt{3}$

11.

$\sqrt{50} = \sqrt{5^2 \cdot 2} = \sqrt{5^2} \sqrt{2} = 5\sqrt{2}$

13.

$\sqrt{245} = \sqrt{7^2 \cdot 5} = \sqrt{7^2} \sqrt{5} = 7\sqrt{5}$

15.

$\sqrt{98} = \sqrt{7^2 \cdot 2} = \sqrt{7^2} \sqrt{2} = 7\sqrt{2}$
17. \[\sqrt{45} = \sqrt{3^2 \cdot 5} = \sqrt{3^2} \sqrt{5} = 3\sqrt{5}\]

19. \[\sqrt{24} = \sqrt{2^3 \cdot 6} = \sqrt{2^3} \sqrt{6} = 2\sqrt{6}\]

21. Note that \(2 + 0 + 1 + 6 = 9\), which is divisible by 9. Thus, 2016 is divisible by 9. Indeed,
\[2016 = 9 \cdot 224.\]
The last two digits of 224 are 24, which is divisible by 4. Thus, 224 is divisible by 4. Indeed, \(224 = 4 \cdot 56\).
\[2016 = 9 \cdot 224 = (3 \cdot 3) \cdot (4 \cdot 56).\]
Continue to primes.
\[2016 = 3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 7 = 2^5 \cdot 3^2 \cdot 7.\]
Factor out a perfect square (exponents must be divisible by 2).
\[\sqrt{2016} = \sqrt{2^5 \cdot 3^2 \cdot 7} = \sqrt{2^4 \cdot 3^2 \sqrt{2} \cdot 7} = 2^2 \cdot 3 \sqrt{2} \cdot 7 = 12\sqrt{14}\]
Checking,

\[12 \cdot \sqrt{14} \\ 12 \cdot 3.872983346 \\ 46.4757961528\]

23. Money! Anything that ends in 00, 25, 50, or 75 is divisible by 25. Indeed, 14175 = 25 \cdot 567. Further, \(5 + 6 + 7 = 18\), so 567 is divisible by 9; i.e., \(567 = 9 \cdot 63\). Continuing to primes,
\[14175 = 25 \cdot 567 = 5 \cdot 5 \cdot 9 \cdot 63 = 5 \cdot 5 \cdot 3 \cdot 3 \cdot 3 \cdot 7.\]
Factor out a perfect square (exponents divisible by 2).
\[\sqrt{14175} = \sqrt{3^4 \cdot 5^2 \cdot 7} = \sqrt{3^4} \cdot 5 \sqrt{7} = 3^2 \cdot 5 \sqrt{7} = 45\sqrt{7}\]

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Checking,

\[
\sqrt{14175} \approx 119.058809
\]

\[
45 \cdot \sqrt{7} \approx 119.058809
\]

25. Money! Anything that ends in 00, 25, 50, or 75 is divisible by 25. Indeed, 20250 = 25 \cdot 810. Continuing to primes,

\[
20250 = 5 \cdot 5 \cdot 9 \cdot 9 \cdot 10 = 5 \cdot 5 \cdot 3 \cdot 3 \cdot 3 \cdot 2 \cdot 5.
\]

Factor out a perfect square.

\[
\sqrt{20250} = \sqrt{2 \cdot 3^4 \cdot 5^3} = \sqrt{3^4 \cdot 5^2 \cdot 2 \cdot 5} = 3^2 \cdot 5 \sqrt{2 \cdot 5} = 45 \sqrt{10}
\]

Checking,

\[
\sqrt{20250} \approx 142.302497
\]

\[
45 \cdot \sqrt{10} \approx 142.302497
\]

27.

\[
\sqrt{(6x - 11)^4} = \sqrt{(6x - 11)^2} = |(6x - 11)^2|
\]

However, \((6x - 11)^2\) is already nonnegative, so the absolute value bars are unnecessary. Hence,

\[
\sqrt{(6x - 11)^4} = (6x - 11)^2.
\]

29.

\[
\sqrt{25f^2} = \sqrt{25} \sqrt{f^2} = 5|f|
\]

Because \(f\) can be any real number, we cannot remove the absolute value bars without more information.

31.

\[
\sqrt{16m^2} = \sqrt{4^2m^2} = \sqrt{4^2} \sqrt{m^2} = 4|m|
\]

Since the index on the radical is even and, after simplification, the variable is raised to an odd power, absolute value signs around the simplified variable are necessary.
33. 
\[ \sqrt{(7x + 5)^{12}} = \sqrt{((7x + 5)^6)^2} = |(7x + 5)^6| \]
However, \((7x + 5)^6\) is already nonnegative, so absolute value signs are unnecessary.
\[ \sqrt{(7x + 5)^{12}} = (7x + 5)^6 \]

35. 
\[ \sqrt{25x^2 - 50x + 25} = \sqrt{(5x - 5)^2} = |5x - 5| \]
Because \(x\) can be any real number, the absolute value signs around the simplified binomial are necessary.

37. 
\[ \sqrt{25x^2 + 90x + 81} = \sqrt{(5x + 9)^2} = |5x + 9| \]
Because \(x\) can be any real number, the absolute value signs around the simplified binomial are necessary.

39. 
\[ \sqrt{(3x + 6)^{12}} = \sqrt{((3x + 6)^6)^2} = |(3x + 6)^6| \]
However, the expression \((3x + 6)^6\) is already nonnegative, so the absolute value bars are unnecessary.
\[ \sqrt{(3x + 6)^{12}} = (3x + 6)^6 \]

41. 
\[ \sqrt{36x^2 + 36x + 9} = \sqrt{(6x + 3)^2} = |6x + 3| \]
Because \(x\) can be any real number, the absolute value signs around the simplified binomial are necessary.

43. 
\[ \sqrt{4p^{10}} = \sqrt{4 \sqrt{(p^5)^2}} = 2|p^5| \]
Now, we can use the multiplicative property of absolute values and write
\[ 2|p^5| = 2|p^4||p| = 2p^4|p|. \]
Since \(p\) can be any real number, absolute value signs around the simplified variable are necessary.
45. 
\[ \sqrt{25q^6} = \sqrt{25(q^3)^2} = 5|q^3| \]

Now, we can use the multiplicative property of absolute values and write
\[ 5|q^3| = 5|q^2||q| = 5q^2|q|. \]

Because \( q \) can be any real number, absolute value signs around the simplified variable are necessary.

47. Factor out a perfect square.
\[ \sqrt{32x^6} = \sqrt{16x^6\sqrt{2}} = \sqrt{16\sqrt{x^6}\sqrt{2}} = 4|x^3|\sqrt{2} \]

However, \(|x^3| = |x^2||x| = x^2|x|, \) since \( x^2 \geq 0 \). Thus,
\[ \sqrt{32x^6} = 4x^2|x|\sqrt{2}. \]

If \( x < 0 \), then \(|x| = -x \) and
\[ \sqrt{32x^6} = 4x^2(-x)\sqrt{2} = -4x^3\sqrt{2}. \]

Checking with \( x = -2 \).

49. Factor out a perfect square.
\[ \sqrt{27x^{12}} = \sqrt{9x^{12}\sqrt{3}} = \sqrt{9\sqrt{x^{12}\sqrt{3}}} = 3|x^6|\sqrt{3}. \]

However, \(|x^6| = x^6 \) since \( x^6 \geq 0 \). Thus,
\[ \sqrt{27x^{12}} = 3x^6\sqrt{3}. \]

Checking with \( x = -2 \).
51. Factor the perfect square trinomial.

\[ \sqrt{x^2 - 8x + 16} = \sqrt{(x - 4)^2} = |x - 4| \]

If \( x < 4 \), or equivalently, if \( x - 4 < 0 \), then \( |x - 4| = -(x - 4) \). Thus,

\[ \sqrt{x^2 - 8x + 16} = -x + 4. \]

In (b), we’ve drawn the graph of \( y = \sqrt{x^2 - 8x + 16} \). In (d), we’ve drawn the graph of \( y = -x + 4 \). Note that the graphs in (b) and (d) agree when \( x < 4 \), lending credence to the fact that \( \sqrt{x^2 - 8x + 16} = -x + 4 \) when \( x < 4 \).

53. Factor the perfect square trinomial.

\[ \sqrt{x^2 - 10x + 25} = \sqrt{(x - 5)^2} = |x - 5| \]

If \( x \geq 5 \), or equivalently, \( x - 5 \geq 0 \), then \( |x - 5| = x - 5 \). Hence,

\[ \sqrt{x^2 - 10x + 25} = x - 5. \quad (1) \]

In (b), we’ve drawn the graph of \( y = \sqrt{x^2 - 10x + 25} \). In (d), we’ve drawn the graph of \( y = x - 5 \). Note that the graphs in (b) and (d) agree when \( x \geq 5 \), lending credence to the fact that \( \sqrt{x^2 - 10x + 25} = x - 5 \) when \( x \geq 5 \).

55.

\[ \sqrt[3]{9d} \cdot \sqrt[3]{12d} \cdot \sqrt[3]{d} = 3d^2 \sqrt[3]{d} \]

57.

\[ \sqrt{25x^2 + 40x + 16} = \sqrt{(5x + 4)^2} = 5x + 4 \]
\[ \sqrt{4j^{11}} = \sqrt{4} \sqrt{j^{10}} \sqrt{j} = 2j^5 \sqrt{j} \]

61.
\[ \sqrt{25m^2} = \sqrt{25} \sqrt{m^2} = 5m \]

63.
\[ \sqrt{4c^5} = \sqrt{4} \sqrt{c^4} \sqrt{c} = 2c^2 \sqrt{c} \]

65.
\[ \sqrt{25h^{10}} = \sqrt{25} \sqrt{h^{10}} = 5h^5 \]

67.
\[ \sqrt{9s^7} = \sqrt{9} \sqrt{s^6} \sqrt{s} = 3s^3 \sqrt{s} \]

69.
\[ \sqrt{4p^8} = \sqrt{4} \sqrt{p^6} = 2p^4 \]

71.
\[ \sqrt{9q^{10}} = \sqrt{9} \sqrt{q^8} = 3q^5 \]

73.
\[ \sqrt{2f^5} \sqrt{8f^3} = \sqrt{2} \cdot 8 \cdot f^5 \cdot f^3 = \sqrt{16f^8} = \sqrt{16} \sqrt{(f^4)^2} = 4f^4 \]

75.
\[ \sqrt{2k^7} \sqrt{32k^3} = \sqrt{2} \cdot 32 \cdot k^7 \cdot k^3 = \sqrt{64k^{10}} = \sqrt{64} \sqrt{(k^5)^2} = 8k^5 \]

77.
\[ \sqrt{2e^9} \sqrt{8e^3} = \sqrt{2} \cdot 8 \cdot e^9 \cdot e^3 = \sqrt{16e^{12}} = \sqrt{16} \sqrt{(e^6)^2} = 4e^6 \]

79.
\[ \sqrt{3z^5} \sqrt{27z^3} = \sqrt{3} \cdot 27 \cdot z^5 \cdot z^3 = \sqrt{81z^8} = \sqrt{81} \sqrt{(z^4)^2} = 9z^4 \]

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9.3 Exercises

1. Use a calculator to first approximate \( \sqrt{5}/\sqrt{2} \). On the same screen, approximate \( \sqrt{5}/2 \). Report the results on your homework paper.

2. Use a calculator to first approximate \( \sqrt{7}/\sqrt{5} \). On the same screen, approximate \( \sqrt{7}/5 \). Report the results on your homework paper.

3. Use a calculator to first approximate \( \sqrt{12}/\sqrt{2} \). On the same screen, approximate \( \sqrt{6} \). Report the results on your homework paper.

4. Use a calculator to first approximate \( \sqrt{15}/\sqrt{5} \). On the same screen, approximate \( \sqrt{3} \). Report the results on your homework paper.

In Exercises 5-16, place each radical expression in simple radical form. As in Example 2 in the narrative, check your result with your calculator.

5. \( \sqrt{\frac{3}{8}} \)

6. \( \sqrt{\frac{5}{12}} \)

7. \( \sqrt{\frac{11}{20}} \)

8. \( \sqrt{\frac{3}{2}} \)

9. \( \sqrt{\frac{11}{18}} \)

10. \( \sqrt{\frac{5}{7}} \)

11. \( \sqrt{\frac{1}{3}} \)

12. \( \sqrt{\frac{16}{5}} \)

13. \( \sqrt{\frac{49}{12}} \)

14. \( \sqrt{\frac{81}{20}} \)

15. \( \sqrt{\frac{100}{7}} \)

16. \( \sqrt{\frac{36}{5}} \)

17. \( \frac{1}{\sqrt{12}} \)

18. \( \frac{1}{\sqrt{8}} \)

19. \( \frac{1}{\sqrt{20}} \)

20. \( \frac{1}{\sqrt{27}} \)

21. \( \frac{6}{\sqrt{8}} \)

22. \( \frac{4}{\sqrt{12}} \)

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23. \( \frac{5}{\sqrt{20}} \)
24. \( \frac{9}{\sqrt{27}} \)
25. \( \frac{6}{2\sqrt{3}} \)
26. \( \frac{10}{3\sqrt{5}} \)
27. \( \frac{15}{2\sqrt{20}} \)
28. \( \frac{3}{2\sqrt{18}} \)

In Exercises 29-36, place the given radical expression in simple form. Use prime factorization as in Example 8 in the narrative to help you with the calculations. As in Example 6, check your result with your calculator.

29. \( \frac{1}{\sqrt{96}} \)
30. \( \frac{1}{\sqrt{432}} \)
31. \( \frac{1}{\sqrt{250}} \)
32. \( \frac{1}{\sqrt{108}} \)
33. \( \sqrt{\frac{5}{96}} \)
34. \( \sqrt{\frac{2}{135}} \)
35. \( \sqrt{\frac{2}{1485}} \)
36. \( \sqrt{\frac{3}{280}} \)

In Exercises 37-44, place each of the given radical expressions in simple radical form. Make no assumptions about the sign of any variable. Variables can represent either positive or negative numbers.

37. \( \sqrt{\frac{8}{x^4}} \)
38. \( \sqrt{\frac{12}{x^6}} \)
39. \( \sqrt{\frac{20}{x^2}} \)
40. \( \sqrt{\frac{32}{x^{14}}} \)
41. \( \sqrt[2]{\frac{2}{8x^5}} \)
42. \( \sqrt[3]{\frac{3}{12x^6}} \)
43. \( \frac{10}{\sqrt{20x^{10}}} \)
44. \( \frac{12}{\sqrt{6x^4}} \)

In Exercises 45-48, follow the lead of Example 8 in the narrative to craft a solution.

45. Given that \( x < 0 \), place the radical expression \( 6/\sqrt{2x^6} \) in simple radical form. Check your solution on your calculator for \( x = -1 \).

46. Given that \( x > 0 \), place the radical expression \( 4/\sqrt{12x^3} \) in simple radical form. Check your solution on your calculator for \( x = 1 \).
47. Given that $x > 0$, place the radical expression $8/\sqrt{8x^5}$ in simple radical form. Check your solution on your calculator for $x = 1$.

48. Given that $x < 0$, place the radical expression $15/\sqrt{20x^6}$ in simple radical form. Check your solution on your calculator for $x = -1$.

In Exercises 49-56, place each of the radical expressions in simple form. Assume that all variables represent positive numbers.

49. $\sqrt{\frac{12}{x}}$

50. $\sqrt{\frac{18}{x}}$

51. $\sqrt{\frac{50}{x^3}}$

52. $\sqrt{\frac{72}{x^5}}$

53. $\frac{1}{\sqrt{50x}}$

54. $\frac{2}{\sqrt{18x}}$

55. $\frac{3}{\sqrt{27x^3}}$

56. $\frac{5}{\sqrt{10x^5}}$
9.3 Solutions

1. Both $\sqrt{5}/\sqrt{2} = \sqrt{5/2} \approx 1.58113883$.

3. Both $\sqrt{12}/\sqrt{2} = \sqrt{6} \approx 2.449489743$.

5. 

$$\sqrt{\frac{3}{8}} = \sqrt{\frac{3}{8} \cdot \frac{2}{2}} = \sqrt{\frac{6}{16}} = \frac{\sqrt{6}}{4}$$

7. 

$$\sqrt{\frac{11}{20}} = \sqrt{\frac{11}{20} \cdot \frac{5}{5}} = \sqrt{\frac{55}{100}} = \frac{\sqrt{55}}{10}$$
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9.
\[ \sqrt{\frac{11}{18}} = \sqrt{\frac{11 \cdot 2}{18 \cdot 2}} = \sqrt{\frac{22}{36}} = \frac{\sqrt{22}}{6} \]

11.
\[ \sqrt{\frac{4}{3}} = \sqrt{\frac{4 \cdot 3}{2 \cdot 3}} = \sqrt{\frac{12}{9}} = \frac{\sqrt{12}}{3} = \frac{\sqrt{4\sqrt{3}}}{3} = \frac{2\sqrt{3}}{3} \]

13.
\[ \sqrt{\frac{49}{12}} = \sqrt{\frac{49 \cdot 3}{12 \cdot 3}} = \sqrt{\frac{49 \cdot 3}{36}} = \frac{\sqrt{49\sqrt{3}}}{6} = \frac{7\sqrt{3}}{6} \]

15.
\[ \sqrt{\frac{100}{7}} = \sqrt{\frac{100 \cdot 7}{7 \cdot 7}} = \sqrt{\frac{100 \cdot 7}{49}} = \frac{\sqrt{100\sqrt{7}}}{7} = \frac{10\sqrt{7}}{7} \]
17. \[
\frac{1}{\sqrt{12}} = \frac{1}{\sqrt{12}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{36}} = \frac{\sqrt{3}}{6}
\]

19. \[
\frac{1}{\sqrt{20}} = \frac{1}{\sqrt{20}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{100}} = \frac{\sqrt{5}}{10}
\]

21. \[
\frac{6}{\sqrt{8}} = \frac{6}{\sqrt{8}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{\sqrt{16}} = \frac{6\sqrt{2}}{4} = \frac{3\sqrt{2}}{2}
\]

23. \[
\frac{5}{\sqrt{20}} = \frac{5}{\sqrt{20}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{5\sqrt{5}}{\sqrt{100}} = \frac{5\sqrt{5}}{10} = \frac{\sqrt{5}}{2}
\]
25. \[
\frac{6}{2\sqrt{3}} = \frac{6}{2} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}}{2\sqrt{9}} = \frac{6\sqrt{3}}{6} = \sqrt{3}
\]

27. \[
\frac{15}{2\sqrt{20}} = \frac{15}{2\sqrt{20}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{15\sqrt{5}}{2\sqrt{100}} = \frac{15\sqrt{5}}{20} = \frac{3\sqrt{5}}{4}
\]

29. \[
\frac{1}{\sqrt{96}} = \frac{1}{\sqrt{2^5 \cdot 3}} = \frac{1}{\sqrt{2^5} \cdot \sqrt{3}} = \frac{\sqrt{2} \cdot \sqrt{3}}{\sqrt{2^6} \cdot \sqrt{3}} = \frac{\sqrt{2} \cdot \sqrt{3}}{2^3 \cdot \sqrt{3}} = \frac{\sqrt{6}}{24}
\]

31. \[
\frac{1}{\sqrt{250}} = \frac{1}{\sqrt{2^2 \cdot 5^3}} = \frac{1}{\sqrt{2^2} \cdot \sqrt{5^3}} = \frac{\sqrt{2} \cdot \sqrt{5}}{\sqrt{2^2} \cdot \sqrt{5^4}} = \frac{\sqrt{2} \cdot \sqrt{5}}{2 \cdot 5^2} = \frac{\sqrt{10}}{50}
\]
33. 
\[ \sqrt{\frac{5}{96}} = \sqrt{\frac{5}{2^6 \cdot 3}} = \sqrt{\frac{5 \cdot 2 \cdot 3}{2^6 \cdot 3}} = \frac{\sqrt{2 \cdot 3 \cdot 5}}{2^3 \cdot 3} = \frac{\sqrt{30}}{24} \]

35. 
\[ \sqrt{\frac{2}{1485}} = \sqrt{\frac{2}{3^3 \cdot 5 \cdot 11}} = \sqrt{\frac{2 \cdot 3 \cdot 5 \cdot 11}{3^3 \cdot 5^2 \cdot 11^2}} = \frac{\sqrt{2 \cdot 3 \cdot 5 \cdot 11}}{3^2 \cdot 5 \cdot 11} = \frac{\sqrt{330}}{494} \]

37. 
\[ \sqrt{x^4} = \frac{\sqrt{8}}{\sqrt{x^4}} = \frac{\sqrt{4\sqrt{2}}}{|x^2|} = \frac{2\sqrt{2}}{x^2} \]

39. 
\[ \sqrt{\frac{20}{x^2}} = \frac{\sqrt{20}}{\sqrt{x^2}} = \frac{\sqrt{4\sqrt{5}}}{|x|} = \frac{2\sqrt{5}}{|x|} \]

41. 
\[ \frac{2}{\sqrt{8x^8}} = \frac{2}{\sqrt{8x^8}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{16x^8}} = \frac{2\sqrt{2}}{|4x^4|} = \frac{2\sqrt{2}}{4x^4} \]

43. 
\[ \frac{10}{\sqrt{20x^{10}}} = \frac{10}{\sqrt{20x^{10}}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{10\sqrt{5}}{\sqrt{100x^{10}}} = \frac{10\sqrt{5}}{|10x^5|} \]

However, \(|10x^5| = |10||x^4||x| = 10x^4|x|\), so 
\[ \frac{10}{\sqrt{20x^{10}}} = \frac{10\sqrt{5}}{10x^4|x|} = \frac{\sqrt{5}}{x^4|x|}. \]
45.

\[
\frac{6}{\sqrt{2x^6}} = \frac{6}{\sqrt{2x^6}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{\sqrt{4x^6}} = \frac{6\sqrt{2}}{2x^3}.
\]

However, \(|2x^3| = |2|x^2||x| = 2x^2|x|\), so

\[
\frac{6\sqrt{2}}{2x^3} = \frac{6\sqrt{2}}{2x^2|x|} = \frac{3\sqrt{2}}{x^2|x|}.
\]

If \(x < 0\), then \(|x| = -x\) and

\[
\frac{3\sqrt{2}}{x^2|x|} = \frac{3\sqrt{2}}{x^2(-x)} = -\frac{3\sqrt{2}}{x^3}.
\]

Checking \(x = -1\).

47.

\[
\frac{8}{\sqrt{8x^5}} = \frac{8}{\sqrt{8x^5}} \cdot \frac{\sqrt{2x}}{\sqrt{2x}} = \frac{8\sqrt{2x}}{\sqrt{16x^6}} = \frac{8\sqrt{2x}}{4x^3}.
\]

However, \(|4x^3| = |4||x^2||x| = 4x^2|x|\), so

\[
\frac{8\sqrt{2x}}{4x^3} = \frac{8\sqrt{2x}}{4x^2|x|} = \frac{2\sqrt{2x}}{x^2|x|}.
\]

But \(x > 0\), so \(|x| = x\) and

\[
\frac{2\sqrt{2x}}{x^2|x|} = \frac{2\sqrt{2x}}{x^2(x)} = \frac{2\sqrt{2x}}{x^3}.
\]

Checking \(x = 1\).
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49.

\[ \sqrt{\frac{12}{x}} = \sqrt{\frac{12}{x^2}} = \frac{\sqrt{12}}{x} = \frac{2\sqrt{3}}{x} \]

51.

\[ \sqrt{\frac{50}{x^3}} = \sqrt{\frac{50}{x^4}} = \frac{\sqrt{50}}{x^2} = \frac{5\sqrt{2}}{x^2} \]

53.

\[ \frac{1}{\sqrt{50x}} = \frac{\sqrt{2x}}{\sqrt{100x^2}} = \frac{\sqrt{2x}}{10x} \]

55.

\[ \frac{3}{\sqrt{27x^3}} = \frac{3}{\sqrt{27x^4}} = \frac{\sqrt{3x}}{9x^2} = \frac{\sqrt{3x}}{3x^2} \]
9.4 Exercises

In Exercises 1-14, place each of the radical expressions in simple radical form. Check your answer with your calculator.

1. $2(5\sqrt{7})$
2. $-3(2\sqrt{3})$
3. $-\sqrt{3}(2\sqrt{5})$
4. $\sqrt{2}(3\sqrt{7})$
5. $\sqrt{3}(5\sqrt{6})$
6. $\sqrt{2}(-3\sqrt{10})$
7. $(2\sqrt{5})(-3\sqrt{3})$
8. $(-5\sqrt{2})(-2\sqrt{7})$
9. $(-4\sqrt{3})(2\sqrt{6})$
10. $(2\sqrt{5})(-3\sqrt{10})$
11. $(2\sqrt{3})^2$
12. $(-3\sqrt{5})^2$
13. $(-5\sqrt{2})^2$
14. $(7\sqrt{11})^2$

In Exercises 23-30, combine like terms. Place your final answer in simple radical form. Check your solution with your calculator.

18. $-3(4 - 3\sqrt{2})$
19. $\sqrt{2}(2 + \sqrt{2})$
20. $\sqrt{3}(4 - \sqrt{6})$
21. $\sqrt{2}(\sqrt{10} + \sqrt{14})$
22. $\sqrt{3}(\sqrt{15} - \sqrt{33})$

In Exercises 15-22, use the distributive property to multiply. Place your final answer in simple radical form. Check your result with your calculator.

15. $2(3 + \sqrt{5})$
16. $-3(4 - \sqrt{7})$
17. $2(-5 + 4\sqrt{2})$
18. $-3(4 - 3\sqrt{2})$
19. $\sqrt{2}(2 + \sqrt{2})$
20. $\sqrt{3}(4 - \sqrt{6})$
21. $\sqrt{2}(\sqrt{10} + \sqrt{14})$
22. $\sqrt{3}(\sqrt{15} - \sqrt{33})$
23. $-5\sqrt{2} + 7\sqrt{2}$
24. $2\sqrt{3} + 3\sqrt{3}$
25. $2\sqrt{6} - 8\sqrt{6}$
26. $\sqrt{7} - 3\sqrt{7}$
27. $2\sqrt{3} - 4\sqrt{2} + 3\sqrt{3}$
28. $7\sqrt{5} + 2\sqrt{7} - 3\sqrt{5}$
29. $2\sqrt{3} + 5\sqrt{2} - 7\sqrt{3} + 2\sqrt{2}$
30. $3\sqrt{11} - 2\sqrt{7} - 2\sqrt{11} + 4\sqrt{7}$

In Exercises 31-40, combine like terms where possible. Place your final answer in simple radical form. Use your calculator to check your result.

31. $\sqrt{45} + \sqrt{20}$
32. $-4\sqrt{45} - 4\sqrt{20}$
33. $2\sqrt{18} - \sqrt{8}$

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34. $-\sqrt{20} + 4\sqrt{45}$
35. $-5\sqrt{27} + 5\sqrt{12}$
36. $3\sqrt{12} - 2\sqrt{27}$
37. $4\sqrt{20} + 4\sqrt{45}$
38. $-2\sqrt{18} - 5\sqrt{8}$
39. $2\sqrt{45} + 5\sqrt{20}$
40. $3\sqrt{27} - 4\sqrt{12}$

In Exercises 49-60, multiply to expand each of the given radical expressions. Place your final answer in simple radical form. Use your calculator to check your result.

49. $(2 + \sqrt{3})(3 - \sqrt{3})$
50. $(5 + \sqrt{2})(2 - \sqrt{2})$
51. $(4 + 3\sqrt{2})(2 - 5\sqrt{2})$
52. $(3 + 5\sqrt{3})(1 - 2\sqrt{3})$
53. $(2 + 3\sqrt{2})(2 - 3\sqrt{2})$
54. $(3 + 2\sqrt{5})(3 - 2\sqrt{5})$
55. $(2\sqrt{3} + 3\sqrt{2})(2\sqrt{3} - 3\sqrt{2})$
56. $(8\sqrt{2} + \sqrt{5})(8\sqrt{2} - \sqrt{5})$
57. $(2 + \sqrt{3})^2$
58. $(3 - \sqrt{2})^2$
59. $(\sqrt{3} - 2\sqrt{5})^2$
60. $(2\sqrt{3} + 3\sqrt{2})^2$

In Exercises 61-68, place each of the given rational expressions in simple radical form by “rationalizing the denominator.” Check your result with your calculator.

61. $\frac{1}{\sqrt{3} + \sqrt{3}}$
62. $\frac{1}{2\sqrt{3} - \sqrt{2}}$
63. $\frac{6}{2\sqrt{5} - \sqrt{2}}$
64. $\frac{9}{3\sqrt{3} - \sqrt{6}}$
65. \[ \frac{2 + \sqrt{3}}{2 - \sqrt{3}} \]
66. \[ \frac{3 - \sqrt{5}}{3 + \sqrt{5}} \]
67. \[ \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}} \]
68. \[ \frac{2\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}} \]

In **Exercises 69-76**, use the quadratic formula to find the solutions of the given equation. Place your solutions in simple radical form and reduce your solutions to lowest terms.

69. \[ 3x^2 - 8x = 5 \]
70. \[ 5x^2 - 2x = 1 \]
71. \[ 5x^2 = 2x + 1 \]
72. \[ 3x^2 - 2x = 11 \]
73. \[ 7x^2 = 6x + 2 \]
74. \[ 11x^2 + 6x = 4 \]
75. \[ x^2 = 2x + 19 \]
76. \[ 100x^2 = 40x - 1 \]

78. Given \( f(x) = \sqrt{x + 2} \), evaluate the expression
\[ \frac{f(x) - f(3)}{x - 3}, \]
and then “rationalize the numerator.”

79. Given \( f(x) = \sqrt{x} \), evaluate the expression
\[ \frac{f(x + h) - f(x)}{h}, \]
and then “rationalize the numerator.”

80. Given \( f(x) = \sqrt{x - 3} \), evaluate the expression
\[ \frac{f(x + h) - f(x)}{h}, \]
and then “rationalize the numerator.”

In **Exercises 77-80**, we will suspend the usual rule that you should rationalize the denominator. Instead, just this one time, rationalize the numerator of the resulting expression.

77. Given \( f(x) = \sqrt{x} \), evaluate the expression
\[ \frac{f(x) - f(2)}{x - 2}, \]
and then “rationalize the numerator.”
9.4 Solutions

1. Regroup using the associative property and simplify.

\[ 2(5\sqrt{7}) = (2 \cdot 5)\sqrt{7} = 10\sqrt{7} \]

Check.

3. The commutative and associative properties allow us to reorder and regroup.

\[ -\sqrt{3}(2\sqrt{5}) = 2(-\sqrt{3}\sqrt{5}) = 2(-\sqrt{15}) = -2\sqrt{15} \]

Check.

5. The commutative and associative properties allow us to reorder and regroup.

\[ \sqrt{3}(5\sqrt{6}) = 5(\sqrt{3}\sqrt{6}) = 5\sqrt{18} \]

This is not in simple form as it is possible to factor out a perfect square.

\[ 5\sqrt{18} = 5\sqrt{9}\sqrt{2} = 5 \cdot 3\sqrt{2} = 15\sqrt{2} \]

Check.
7. The commutative and associative properties allow us to reorder and regroup.
\[(2\sqrt{5})(-3\sqrt{3}) = (2 \cdot -3)(\sqrt{5}\sqrt{3}) = -6\sqrt{15}\]
Check.

9. The commutative and associative properties allows us to reorder and regroup.
\[(-4\sqrt{3})(2\sqrt{6}) = (-4 \cdot 2)(\sqrt{3}\sqrt{6}) = -8\sqrt{18}\]
This answer is not in simple form because we can factor out a perfect square.
\[-8\sqrt{18} = -8\sqrt{9}\sqrt{2} = -8 \cdot 3\sqrt{2} = -24\sqrt{2}\]
Check.

11. Recall that \((ab)^2 = a^2b^2\).
\[(2\sqrt{3})^2 = (2)^2(\sqrt{3})^2 = 4 \cdot 3 = 12\]
Check.
13. Recall that \((ab)^2 = a^2b^2\).

\[
(-5\sqrt{2})^2 = (-5)^2(\sqrt{2})^2 = 25 \cdot 2 = 50
\]

Check.

15. Recall the distributive property: \(a(b + c) = ab + ac\).

\[
2(3 + \sqrt{5}) = 2(3) + 2(\sqrt{5}) = 6 + 2\sqrt{5}
\]

Check.

17. Recall the distributive property: \(a(b + c) = ab + ac\).

\[
2(-5 + 4\sqrt{2}) = 2(-5) + 2(4\sqrt{2}) = -10 + 8\sqrt{2}
\]

Check.

19. Use the distributive property: \(a(b + c) = ab + ac\).

\[
\sqrt{2}(2 + \sqrt{2}) = \sqrt{2}(2) + \sqrt{2}(\sqrt{2}) = 2\sqrt{2} + \sqrt{4} = 2\sqrt{2} + 2
\]

Check.
21. Use the distributive property: \(a(b + c) = ab + ac\)

\[
\sqrt{2}(\sqrt{10} + \sqrt{14}) = \sqrt{2}(\sqrt{10}) + \sqrt{2}(\sqrt{14}) = \sqrt{20} + \sqrt{28}
\]

However, this answer is not in simple form because we can factor out perfect squares.

\[
\sqrt{20} + \sqrt{28} = \sqrt{4\sqrt{5}} + \sqrt{4\sqrt{7}} = 2\sqrt{5} + 2\sqrt{7}
\]

Check.

23. Use the distributive property to factor out \(\sqrt{2}\).

\[
-5\sqrt{2} + 7\sqrt{2} = (-5 + 7)\sqrt{2} = 2\sqrt{2}
\]

In practice, we usually just combine \(-5\sqrt{2} + 7\sqrt{2}\) much as we do \(-5x + 7x = 2x\) and simply write \(-5\sqrt{2} + 7\sqrt{2} = 2\sqrt{2}\).

Check.

25. Use the distributive property to factor out \(\sqrt{6}\).

\[
2\sqrt{6} - 8\sqrt{6} = (2 - 8)\sqrt{6} = -6\sqrt{6}
\]

In practice, we usually just combine \(2\sqrt{6} - 8\sqrt{6}\) much as we do \(2x - 8x = -6x\) and simply write \(2\sqrt{6} - 8\sqrt{6} = -6\sqrt{6}\).

Check.
27. The commutative and associative properties of addition allows us to reorder and regroup, then we combine like terms.

\[ 2\sqrt{3} - 4\sqrt{2} + 3\sqrt{3} = (2\sqrt{3} + 3\sqrt{3}) - 4\sqrt{2} = 5\sqrt{3} - 4\sqrt{2} \]

Check.

29. The commutative and associative properties of addition allow us to reorder and regroup, then we can add like terms.

\[ 2\sqrt{3} + 5\sqrt{2} - 7\sqrt{3} + 2\sqrt{2} = (2\sqrt{3} - 7\sqrt{3}) + (5\sqrt{2} + 2\sqrt{2}) = -5\sqrt{3} + 7\sqrt{2} \]

Check.

31.

\[ \sqrt{45} + \sqrt{20} = \sqrt{3^2 \cdot 5} + \sqrt{2^2 \cdot 5} \]
\[ = 3\sqrt{5} + 2\sqrt{5} \]
\[ = (3 + 2)\sqrt{5} \]
\[ = 5\sqrt{5} \]

Check.
33. \[2\sqrt{18} - \sqrt{8} = 2\sqrt{3^2 \cdot 2} - \sqrt{2^2 \cdot 2} = 6\sqrt{2} - 2\sqrt{2} = (6-2)\sqrt{2} = 4\sqrt{2}\]

Check.

35. \[-5\sqrt{27} + 5\sqrt{12} = -5\sqrt{3^2 \cdot 3} + 5\sqrt{2^2 \cdot 3} = -15\sqrt{3} + 10\sqrt{3} = (-15 + 10)\sqrt{3} = -5\sqrt{3}\]

Check.

37. \[4\sqrt{20} + 4\sqrt{45} = 4\sqrt{2^2 \cdot 5} + 4\sqrt{3^2 \cdot 5} = 8\sqrt{5} + 12\sqrt{5} = (8+12)\sqrt{5} = 20\sqrt{5}\]
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Check.

\[
2\sqrt{45} + 5\sqrt{20} = 2\sqrt{3^2 \cdot 5} + 5\sqrt{2^2 \cdot 5} \\
= 6\sqrt{5} + 10\sqrt{5} \\
= (6 + 10)\sqrt{5} \\
= 16\sqrt{5}
\]

Check.

41. Place the second term in simple radical form.

\[
\sqrt{2} - \frac{1}{\sqrt{2}} = \sqrt{2} - \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2} - \frac{\sqrt{2}}{2} = \sqrt{2} - \frac{\sqrt{2}}{2}
\]

Write each term over a common denominator of 2.

\[
\sqrt{2} - \frac{\sqrt{2}}{2} = \sqrt{2} \cdot \frac{2}{2} - \frac{\sqrt{2}}{2} = \frac{2\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}
\]

Check.

43. Place the second term in simple radical form.

\[
2\sqrt{2} - \frac{2}{\sqrt{2}} = 2\sqrt{2} - \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = 2\sqrt{2} - \frac{2\sqrt{2}}{\sqrt{4}}
\]

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Continuing,

\[ 2\sqrt{2} - \frac{2\sqrt{2}}{4} = 2\sqrt{2} - \frac{\sqrt{2}}{2} = 2\sqrt{2} - \sqrt{2} = \sqrt{2}. \]

Check.

45. Place the second term in simple radical form.

\[ 5\sqrt{2} + \frac{3}{\sqrt{2}} = 5\sqrt{2} + \frac{3\sqrt{2}}{\sqrt{2}} = 5\sqrt{2} + \frac{3\sqrt{2}}{\sqrt{4}} = 5\sqrt{2} + \frac{3\sqrt{2}}{2} \]

Write equivalent fractions with a common denominator and add.

\[ 5\sqrt{2} + \frac{3\sqrt{2}}{2} = 5\sqrt{2} \cdot \frac{2}{2} + \frac{3\sqrt{2}}{2} = \frac{10\sqrt{2}}{2} \]

Check.

47. Place the first and second terms in simple radical form.

\[ \sqrt{8} - \frac{12}{\sqrt{2}} - 3\sqrt{2} = \sqrt{4\sqrt{2}} - \frac{12\sqrt{2}}{\sqrt{2}} - 3\sqrt{2} = 2\sqrt{2} - \frac{12\sqrt{2}}{2} - 3\sqrt{2} \]

Reduce the fractional second term, then combine like terms.

\[ 2\sqrt{2} - \frac{12\sqrt{2}}{2} - 3\sqrt{2} = 2\sqrt{2} - 6\sqrt{2} - 3\sqrt{2} = -7\sqrt{2} \]

Check.
49. Distribute the second factor times each term of the first factor, then apply the distributive property a second time.

\[(2 + \sqrt{3})(3 - \sqrt{3}) = 2(3 - \sqrt{3}) + \sqrt{3}(3 - \sqrt{3}) = 6 - 2\sqrt{3} + 3\sqrt{3} - \sqrt{9}\]

Simplify and combine like terms.

\[6 - 2\sqrt{3} + 3\sqrt{3} - 3 = 3 + \sqrt{3}\]

Check.

\[(2 + \sqrt{3})(3 - \sqrt{3}) = 4.732050808\]

51. Use the distributive property to multiply the second factor times each term of the first factor, then use the distributive property a second time.

\[(4 + 3\sqrt{2})(2 - 5\sqrt{2}) = 4(2 - 5\sqrt{2}) + 3\sqrt{2}(2 - 5\sqrt{2}) = 8 - 20\sqrt{2} + 6\sqrt{2} - 15\sqrt{4}\]

Simplify, then combine like terms.

\[8 - 20\sqrt{2} + 6\sqrt{2} - 15\sqrt{4} = 8 - 20\sqrt{2} + 6\sqrt{2} - 30 = -22 - 14\sqrt{2}\]

Check.

\[(4 + 3\sqrt{2})(2 - 5\sqrt{2}) = -41.79898987\]

53. Here we use the difference of squares pattern: \((a + b)(a - b) = a^2 - b^2\).

\[(2 + 3\sqrt{2})(2 - 3\sqrt{2}) = (2)^2 - (3\sqrt{2})^2\]

Recall that \((ab)^2 = a^2b^2\).

\[(2)^2 - (3\sqrt{2})^2 = 4 - (3)^2(\sqrt{2})^2 = 4 - 9 \cdot 2 = 4 - 18 = -14\]

Check.

\[(2 + 3\sqrt{2})(2 - 3\sqrt{2}) = -14\]

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55. Here we use the difference of squares pattern: 
\[(a + b)(a - b) = a^2 - b^2.\]
\[(2\sqrt{3} + 3\sqrt{2})(2\sqrt{3} - 3\sqrt{2}) = (2\sqrt{3})^2 - (3\sqrt{2})^2\]
Recall that \((ab)^2 = a^2b^2\).
\[(2\sqrt{3})^2 - (3\sqrt{2})^2 = (2)^2(\sqrt{3})^2 - (3)^2(\sqrt{2})^2 = 4 \cdot 3 - 9 \cdot 2 = 12 - 18 = -6\]
Check.

57. Here we use the squaring a binomial pattern: 
\[(a + b)^2 = a^2 + 2ab + b^2.\]
\[(2 + \sqrt{5})^2 = (2)^2 + 2(2)(\sqrt{5}) + (\sqrt{5})^2 = 4 + 4\sqrt{5} + 5 = 9 + 4\sqrt{5}\]
Check.

59. Here we use the squaring a binomial pattern: 
\[(a - b)^2 = a^2 - 2ab + b^2.\] Again, recall that \((ab)^2 = a^2b^2\).
\[(\sqrt{3} - 2\sqrt{5})^2 = (\sqrt{3})^2 - 2(\sqrt{3})(2\sqrt{5}) + (2\sqrt{5})^2 = (\sqrt{3})^2 - 2(\sqrt{3})(2\sqrt{5}) + (2)^2(\sqrt{5})^2\]
Continuing.
\[(\sqrt{3})^2 - 2(\sqrt{3})(2\sqrt{5}) + (2)^2(\sqrt{5})^2 = 3 - 4\sqrt{15} + 4 \cdot 5 = 3 - 4\sqrt{15} + 20 = 23 - 4\sqrt{15}\]
Check.
61. Multiply numerator and denominator by \( \sqrt{5} - \sqrt{3} \). Recall the difference of squares pattern: \((a + b)(a - b) = a^2 - b^2\).

\[
\frac{1}{\sqrt{5} + \sqrt{3}} = \frac{1}{\sqrt{5} + \sqrt{3}} \cdot \frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} - \sqrt{3}} = \frac{\sqrt{5} - \sqrt{3}}{(\sqrt{5})^2 - (\sqrt{3})^2}
\]

Continuing.

\[
\frac{\sqrt{5} - \sqrt{3}}{(\sqrt{5})^2 - (\sqrt{3})^2} = \frac{\sqrt{5} - \sqrt{3}}{5 - 3} = \frac{\sqrt{5} - \sqrt{3}}{2}
\]

Check.

63. Multiply numerator and denominator by \( 2\sqrt{5} + \sqrt{2} \). Recall the difference of squares pattern: \((a + b)(a - b) = a^2 - b^2\).

\[
\frac{6}{2\sqrt{5} - \sqrt{2}} = \frac{6}{2\sqrt{5} - \sqrt{2}} \cdot \frac{2\sqrt{5} + \sqrt{2}}{2\sqrt{5} + \sqrt{2}} = \frac{12\sqrt{5} + 6\sqrt{2}}{(2\sqrt{5})^2 - (\sqrt{2})^2}
\]

Continuing.

\[
\frac{12\sqrt{5} + 6\sqrt{2}}{(2\sqrt{5})^2 - (\sqrt{2})^2} = \frac{12\sqrt{5} + 6\sqrt{2}}{4 \cdot 5 - 2} = \frac{12\sqrt{5} + 6\sqrt{2}}{20 - 2} = \frac{12\sqrt{5} + 6\sqrt{2}}{18}
\]

Reduce. Factor the numerator and denominator and cancel.

\[
\frac{12\sqrt{5} + 6\sqrt{2}}{18} = \frac{6(2\sqrt{5} + \sqrt{2})}{6 \cdot 3} = \frac{\cancel{6}(2\sqrt{5} + \sqrt{2})}{\cancel{6} \cdot 3} = \frac{2\sqrt{5} + \sqrt{2}}{3}
\]

Alternatively, some like to reduce by dividing numerator and denominator by 6.

\[
\frac{12\sqrt{5} + 6\sqrt{2}}{18} = \frac{\frac{12\sqrt{5}}{6} + \frac{6\sqrt{2}}{6}}{\frac{18}{6}} = \frac{2\sqrt{5} + \sqrt{2}}{3}
\]

Check.
65. Multiply numerator and denominator by $2 + \sqrt{3}$.

\[
\frac{2 + \sqrt{3}}{2 - \sqrt{3}} = \frac{2 + \sqrt{3}}{2} \cdot \frac{2 + \sqrt{3}}{2 - \sqrt{3}} = \frac{(2 + \sqrt{3})^2}{(2 - \sqrt{3})(2 + \sqrt{3})}
\]

Use the squaring a binomial pattern $(a + b)^2 = a^2 + 2ab + b^2$ on the numerator and the difference of squares pattern $(a + b)(a - b) = a^2 - b^2$ on the denominator.

\[
\frac{(2 + \sqrt{3})^2}{(2 - \sqrt{3})(2 + \sqrt{3})} = \frac{(2)^2 + 2(2)(\sqrt{3}) + (\sqrt{3})^2}{(2)^2 - (\sqrt{3})^2}
\]

Continuing.

\[
\frac{(2)^2 + 2(2)(\sqrt{3}) + (\sqrt{3})^2}{(2)^2 - (\sqrt{3})^2} = \frac{4 + 4\sqrt{3} + 3}{4 - 3} = 7 + 4\sqrt{3}
\]

Check.

\[
\begin{array}{c}
(2 + \sqrt{3}) / (2 - \sqrt{3}) = 7 + 4\sqrt{3}
\end{array}
\]

67. Multiply numerator and denominator by $\sqrt{3} + \sqrt{2}$.

\[
\frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}} = \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}} \cdot \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2}} = \frac{(\sqrt{3} + \sqrt{2})^2}{(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})}
\]

Use the squaring a binomial pattern $(a + b)^2 = a^2 + 2ab + b^2$ on the numerator and the difference of squares pattern $(a + b)(a - b) = a^2 - b^2$ on the denominator.

\[
\frac{(\sqrt{3} + \sqrt{2})^2}{(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})} = \frac{(\sqrt{3})^2 + 2(\sqrt{3})(\sqrt{2}) + (\sqrt{2})^2}{(\sqrt{3})^2 - (\sqrt{2})^2}
\]

Continuing.

\[
\frac{(\sqrt{3})^2 + 2(\sqrt{3})(\sqrt{2}) + (\sqrt{2})^2}{(\sqrt{3})^2 - (\sqrt{2})^2} = \frac{3 + 2\sqrt{6} + 2}{3 - 2} = 5 + 2\sqrt{6}
\]

Check.

\[
\begin{array}{c}
(\sqrt{3} + \sqrt{2}) / (\sqrt{3} - \sqrt{2}) = 5 + 2\sqrt{6}
\end{array}
\]

69. The equation is nonlinear, so make one side zero.
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\[ 3x^2 - 8x - 5 = 0 \]

Compare \(3x^2 - 8x - 5 = 0\) with \(ax^2 + bx + c = 0\) and note that \(a = 3\), \(b = -8\), and \(c = -5\). Write down the quadratic formula and substitute.

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(-5)}}{2(3)} = \frac{8 \pm \sqrt{124}}{6}
\]

Factor a perfect square from the radical in the numerator.

\[
x = \frac{8 \pm \sqrt{4 \cdot 31}}{6} = \frac{2 \cdot 4 \pm \sqrt{2 \cdot 31}}{2 \cdot 3} = \frac{2(4 \pm \sqrt{31})}{2 \cdot 3} = \frac{4 \pm \sqrt{31}}{3}
\]

71. The equation is nonlinear, so make one side zero.

\[ 5x^2 - 2x - 1 = 0 \]

Compare \(5x^2 - 2x - 1 = 0\) with \(ax^2 + bx + c = 0\) and note that \(a = 5\), \(b = -2\), and \(c = -1\). Write down the quadratic formula and substitute.

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(5)(-1)}}{2(5)} = \frac{2 \pm \sqrt{24}}{10}
\]

Factor a perfect square from the radical in the numerator.

\[
x = \frac{2 \pm \sqrt{4 \cdot 6}}{10} = \frac{2 \cdot 1 \pm \sqrt{2 \cdot 3}}{2 \cdot 5} = \frac{2(1 \pm \sqrt{6})}{2 \cdot 5} = \frac{1 \pm \sqrt{6}}{5}
\]

73. The equation is nonlinear, so make one side zero.

\[ 7x^2 - 6x - 2 = 0 \]

Compare \(7x^2 - 6x - 2 = 0\) with \(ax^2 + bx + c = 0\) and note that \(a = 7\), \(b = -6\), and \(c = -2\). Write down the quadratic formula and substitute.

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(7)(-2)}}{2(7)} = \frac{6 \pm \sqrt{92}}{14}
\]

Factor a perfect square from the radical in the numerator.

\[
x = \frac{6 \pm \sqrt{92}}{14} = \frac{6 \pm \sqrt{4 \cdot 23}}{14} = \frac{6 \pm 2 \sqrt{23}}{14}
\]

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Factor the numerator and cancel.

\[ x = \frac{6 \pm 2\sqrt{23}}{14} = \frac{2(3 \pm \sqrt{23})}{2 \cdot 7} = \frac{2(3 \pm \sqrt{23})}{2 \cdot 7} = \frac{3 \pm \sqrt{23}}{7} \]

75. The equation is nonlinear, so make one side zero.

\[ x^2 - 2x - 19 = 0 \]

Compare \( x^2 - 2x - 19 = 0 \) with \( ax^2 + bx + c = 0 \) and note that \( a = 1 \), \( b = -2 \), and \( c = -19 \). Write down the quadratic formula and substitute.

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(2) \pm \sqrt{(-2)^2 - 4(1)(-19)}}{2(1)} = \frac{2 \pm \sqrt{80}}{2} \]

Factor a perfect square from the radical in the numerator.

\[ x = \frac{2 \pm \sqrt{80}}{2} = \frac{2 \pm \sqrt{16\sqrt{5}}}{2} = \frac{2 \pm 4\sqrt{5}}{2} \]

Factor the numerator and cancel.

\[ x = \frac{2 \pm 4\sqrt{5}}{2} = \frac{2(1 \pm 2\sqrt{5})}{2} = \frac{2(1 \pm 2\sqrt{5})}{2} = 1 \pm 2\sqrt{5} \]

77. If \( f(x) = \sqrt{x} \), then

\[ \frac{f(x) - f(2)}{x - 2} = \frac{\sqrt{x} - \sqrt{2}}{x - 2}. \]

To “rationalize the numerator,” multiply numerator and denominator by \( \sqrt{x} + \sqrt{2} \), then use the difference of squares pattern to simplify.

\[ \frac{\sqrt{x} - \sqrt{2}}{x - 2} \cdot \frac{\sqrt{x} + \sqrt{2}}{\sqrt{x} + \sqrt{2}} = \frac{(\sqrt{x})^2 - (\sqrt{2})^2}{(x - 2)(\sqrt{x} + \sqrt{2})} = \frac{x - 2}{(x - 2)(\sqrt{x} + \sqrt{2})} \]

Numerator and denominator are factored, so we can cancel,

\[ \frac{x - 2}{(x - 2)(\sqrt{x} + \sqrt{2})} = \frac{x - 2}{(x - 2)(\sqrt{x} + \sqrt{2})} = \frac{1}{\sqrt{x} + \sqrt{2}}, \]

provided, of course, that \( x \neq 2 \).

79. If \( f(x) = \sqrt{x} \), then

\[ \frac{f(x + h) - f(x)}{h} = \frac{\sqrt{x + h} - \sqrt{x}}{h}. \]

To “rationalize the numerator,” multiply numerator and denominator by \( \sqrt{x + h} + \sqrt{x} \), then use the difference of squares pattern to simplify.

\[ \frac{\sqrt{x + h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}} = \frac{(\sqrt{x + h})^2 - (\sqrt{x})^2}{h(\sqrt{x + h} + \sqrt{x})} = \frac{x + h - x}{h(\sqrt{x + h} + \sqrt{x})} = \frac{h}{h} = 1 \]

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Simplify, then cancel.

\[
\frac{x + h - x}{h(\sqrt{x} + h + \sqrt{x})} = \frac{h}{h(\sqrt{x} + h + \sqrt{x})} = \frac{h}{h(\sqrt{x} + h + \sqrt{x})} = \frac{1}{\sqrt{x} + h + \sqrt{x}}
\]

The result is valid provided \( h \neq 0 \).
9.5 Exercises

For the rational functions in Exercises 1-6, perform each of the following tasks.

i. Load the function $f$ and the line $y = k$ into your graphing calculator. Adjust the viewing window so that all point(s) of intersection of the two graphs are visible in your viewing window.

ii. Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label the graphs with their equations. Remember to draw all lines with a ruler.

iii. Use the intersect utility to determine the coordinates of the point(s) of intersection. Plot the point of intersection on your homework paper and label it with its coordinates.

iv. Solve the equation $f(x) = k$ algebraically. Place your work and solution next to your graph. Do the solutions agree?

1. $f(x) = \sqrt{x + 3}, \quad k = 2$
2. $f(x) = \sqrt{4 - x}, \quad k = 3$
3. $f(x) = \sqrt{7 - 2x}, \quad k = 4$
4. $f(x) = \sqrt{3x + 5}, \quad k = 5$
5. $f(x) = \sqrt{5 + x}, \quad k = 4$
6. $f(x) = \sqrt{4 - x}, \quad k = 5$

In Exercises 7-12, use an algebraic technique to solve the given equation. Check your solutions.

7. $\sqrt{-5x + 5} = 2$

8. $\sqrt{4x + 6} = 7$
9. $\sqrt{6x - 8} = 8$
10. $\sqrt{2x + 4} = 2$
11. $\sqrt{-3x + 1} = 3$
12. $\sqrt{4x + 7} = 3$

For the rational functions in Exercises 13-16, perform each of the following tasks.

i. Load the function $f$ and the line $y = k$ into your graphing calculator. Adjust the viewing window so that all point(s) of intersection of the two graphs are visible in your viewing window.

ii. Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label the graphs with their equations. Remember to draw all lines with a ruler.

iii. Use the intersect utility to determine the coordinates of the point(s) of intersection. Plot the point of intersection on your homework paper and label it with its coordinates.

iv. Solve the equation $f(x) = k$ algebraically. Place your work and solution next to your graph. Do the solutions agree?

13. $f(x) = \sqrt{x + 3} + x, \quad k = 9$
14. $f(x) = \sqrt{x + 6} - x, \quad k = 4$
15. $f(x) = \sqrt{x - 5} - x, \quad k = -7$
16. $f(x) = \sqrt{x + 5} + x, \quad k = 7$

\footnote{Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/}

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In Exercises 17-24, use an algebraic technique to solve the given equation. Check your solutions.

17. \( \sqrt{x+1} + x = 5 \)
18. \( \sqrt{x+8} - x = 8 \)
19. \( \sqrt{x+4} + x = 8 \)
20. \( \sqrt{x+8} - x = 2 \)
21. \( \sqrt{x+5} - x = 3 \)
22. \( \sqrt{x+5} + x = 7 \)
23. \( \sqrt{x+9} - x = 9 \)
24. \( \sqrt{x+7} + x = 5 \)

For the rational functions in Exercises 25-28, perform each of the following tasks.

i. Load the function \( f \) and the line \( y = k \) into your graphing window calculator. Adjust the viewing window so that all point(s) of intersection of the two graphs are visible in your viewing window.

ii. Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label the graphs with their equations. Remember to draw all lines with a ruler.

iii. Use the intersect utility to determine the coordinates of the point(s) of intersection. Plot the point of intersection on your homework paper and label it with its coordinates.

iv. Solve the equation \( f(x) = k \) algebraically. Place your work and solution next to your graph. Do the solutions agree?

25. \( f(x) = \sqrt{x+1} + \sqrt{x+6}, \ k = 7 \)
26. \( f(x) = \sqrt{x+2} + \sqrt{x+9}, \ k = 7 \)

27. \( f(x) = \sqrt{x+2} + \sqrt{3x+4}, \ k = 2 \)
28. \( f(x) = \sqrt{6x+7} + \sqrt{3x+3}, \ k = 1 \)

In Exercises 29-40, use an algebraic technique to solve the given equation. Check your solutions.

29. \( \sqrt{x+4} - \sqrt{x-35} = 1 \)
30. \( \sqrt{x-16} + \sqrt{x+16} = 8 \)
31. \( \sqrt{x-19} + \sqrt{x-6} = 13 \)
32. \( \sqrt{x+31} - \sqrt{x+12} = 1 \)
33. \( \sqrt{x-2} - \sqrt{x-49} = 1 \)
34. \( \sqrt{x+13} + \sqrt{x+8} = 5 \)
35. \( \sqrt{x+27} - \sqrt{x-22} = 1 \)
36. \( \sqrt{x+10} + \sqrt{x+13} = 3 \)
37. \( \sqrt{x+30} - \sqrt{x-38} = 2 \)
38. \( \sqrt{x+36} - \sqrt{x+11} = 1 \)
39. \( \sqrt{x-17} + \sqrt{x+3} = 10 \)
40. \( \sqrt{x+18} + \sqrt{x+13} = 5 \)
1. Using the calculator’s *intersect* utility, we arrive at the following result.

To solve $f(x) = 2$ algebraically, replace $f(x)$ with $\sqrt{x + 3}$, then square both sides of the resulting equation.

\[
\begin{align*}
  f(x) &= 2 \\
  \sqrt{x + 3} &= 2 \\
  (\sqrt{x + 3})^2 &= (2)^2 \\
  x + 3 &= 4 \\
  x &= 1
\end{align*}
\]

Note that this agrees nicely with the graphical solution, but we should still check that our answer is not an extraneous solution. Check $x = 1$ in

\[
\begin{align*}
  \sqrt{x + 3} &= 2 \\
  \sqrt{1 + 3} &= 2 \\
  \sqrt{4} &= 2
\end{align*}
\]

The solution checks!
3. Using the calculator’s **intersect** utility, we arrive at the following result.

![Graph showing the intersection of functions](image)

To solve \( f(x) = 4 \) algebraically, replace \( f(x) \) with \( \sqrt{7 - 2x} \), then square both sides of the resulting equation.

\[
\begin{align*}
  f(x) &= 4 \\
  \sqrt{7 - 2x} &= 4 \\
  (\sqrt{7 - 2x})^2 &= (4)^2
\end{align*}
\]

Continuing,

\[
\begin{align*}
  7 - 2x &= 16 \\
  -2x &= 6 \\
  x &= -\frac{9}{2}
\end{align*}
\]

Note that this agrees nicely with the graphical solution, but we should still check that our answer is not an extraneous solution. Check \( x = -\frac{9}{2} \) in

\[
\begin{align*}
  \sqrt{7 - 2x} &= 4 \\
  \sqrt{7 - 2(-\frac{9}{2})} &= 4 \\
  \sqrt{7 + 9} &= 4 \\
  \sqrt{16} &= 4
\end{align*}
\]

The solution checks!
5. Using the calculator’s intersect utility, we arrive at the following result.

To solve \( f(x) = 4 \) algebraically, replace \( f(x) \) with \( \sqrt{5 + x} \), then square both sides of the resulting equation.

\[
\begin{align*}
  f(x) &= 4 \\
  \sqrt{5 + x} &= 4 \\
  (\sqrt{5 + x})^2 &= (4)^2
\end{align*}
\]

Continuing,

\[
\begin{align*}
  5 + x &= 16 \\
  x &= 11
\end{align*}
\]

Note that this agrees nicely with the graphical solution, but we should still check that our answer is not an extraneous solution. Check \( x = 11 \) in

\[
\begin{align*}
  \sqrt{5 + x} &= 4 \\
  \sqrt{5 + 11} &= 4 \\
  \sqrt{16} &= 4
\end{align*}
\]

The solution checks!

7.

\[
\begin{align*}
  \sqrt{-5x + 5} &= 2 \implies -5x + 5 = 2^2 \implies -5x + 5 = 4 \implies x = \frac{1}{5}
\end{align*}
\]

9.

\[
\begin{align*}
  \sqrt{6x - 8} &= 8 \implies 6x - 8 = 8^2 \implies 6x - 8 = 64 \implies x = 12
\end{align*}
\]

11.

\[
\begin{align*}
  \sqrt{-3x + 1} &= 3 \implies -3x + 1 = 3^2 \implies -3x + 1 = 9 \implies x = -\frac{8}{3}
\end{align*}
\]
13. The calculator’s *intersect* utility provides the following solution.

To solve \( f(x) = 9 \) algebraically, replace \( f(x) \) with \( \sqrt{x + 3} + x \), isolate the radical, then square both sides of the resulting equation.

\[
\begin{align*}
    f(x) &= 9 \\
    \sqrt{x + 3} + x &= 9 \\
    \sqrt{x + 3} &= 9 - x \\
    (\sqrt{x + 3})^2 &= (9 - x)^2 \\
    x + 3 &= 81 - 18x + x^2
\end{align*}
\]

This last equation is nonlinear, so make one side zero and factor.

\[
\begin{align*}
    0 &= x^2 - 19x + 78 \\
    0 &= (x - 6)(x - 13)
\end{align*}
\]

The solution \( x = 6 \) agrees with the graphical solution above and checking,

\[
\begin{align*}
    \sqrt{x + 3} + x &= 9 \\
    \sqrt{6 + 3} + 6 &= 9. \\
    3 + 6 &= 9
\end{align*}
\]

Thus, \( x = 6 \) is a solution. On the other hand, checking \( x = 13 \) reveals

\[
\begin{align*}
    \sqrt{x + 3} + x &= 9 \\
    \sqrt{13 + 3} + 13 &= 9. \\
    4 + 13 &= 9
\end{align*}
\]

Thus, \( x = 13 \) is not a solution.
15. The calculator’s intersect utility provides the following solution.

To solve $f(x) = -7$ algebraically, replace $f(x)$ with $\sqrt{x - 5} - x$, isolate the radical, then square both sides of the resulting equation.

\[
\begin{align*}
  f(x) &= -7 \\
  \sqrt{x - 5} - x &= -7 \\
  \sqrt{x - 5} &= x - 7 \\
  (\sqrt{x - 5})^2 &= (x - 7)^2 \\
  x - 5 &= x^2 - 14x + 49
\end{align*}
\]

This last equation is nonlinear, so make one side zero and factor.

\[
\begin{align*}
  0 &= x^2 - 15x + 54 \\
  0 &= (x - 9)(x - 6)
\end{align*}
\]

The solution $x = 9$ agrees with the graphical solution above and checking,

\[
\begin{align*}
  \sqrt{9 - 5} - 9 &= -7 \\
  \sqrt{2} - 9 &= -7.
\end{align*}
\]

Thus, $x = 9$ is a solution. On the other hand, checking $x = 6$ reveals

\[
\begin{align*}
  \sqrt{6 - 5} - 6 &= -7 \\
  1 - 6 &= -7
\end{align*}
\]

Thus, $x = 6$ is not a solution.
17.

\[ \sqrt{x+1} + x = 5 \]
\[ \Rightarrow \sqrt{x+1} = -x + 5 \]
\[ \Rightarrow x + 1 = (-x + 5)^2 \]
\[ \Rightarrow x + 1 = x^2 - 10x + 25 \]
\[ \Rightarrow x^2 - 11x + 24 = 0 \]

Solving this quadratic equation yields \( x = 8, 3 \). However, 8 does not solve the original equation.

19.

\[ \sqrt{x+4} + x = 8 \]
\[ \Rightarrow \sqrt{x+4} = -x + 8 \]
\[ \Rightarrow x + 4 = (-x + 8)^2 \]
\[ \Rightarrow x + 4 = x^2 - 16x + 64 \]
\[ \Rightarrow x^2 - 17x + 60 = 0 \]

Solving this quadratic equation yields \( x = 12, 5 \). However, 12 does not solve the original equation.

21.

\[ \sqrt{x+5} - x = 3 \]
\[ \Rightarrow \sqrt{x+5} = x + 3 \]
\[ \Rightarrow x + 5 = (x + 3)^2 \]
\[ \Rightarrow x + 5 = x^2 + 6x + 9 \]
\[ \Rightarrow x^2 + 5x + 4 = 0 \]

Solving this quadratic equation yields \( x = -1, -4 \). However, -4 does not solve the original equation.

23.

\[ \sqrt{x+9} - x = 9 \]
\[ \Rightarrow \sqrt{x+9} = x + 9 \]
\[ \Rightarrow x + 9 = (x + 9)^2 \]
\[ \Rightarrow x + 9 = x^2 + 18x + 81 \]
\[ \Rightarrow x^2 + 17x + 72 = 0 \]
Solving this quadratic equation yields \( x = -8, -9 \).

25. The calculator’s intersect utility provides the following solution.

To solve \( f(x) = 7 \) algebraically, replace \( f(x) \) with \( \sqrt{x-1} + \sqrt{x+6} \), isolate one of the radicals, then square both sides of the resulting equation.

\[
\begin{align*}
f(x) &= 7 \\
\sqrt{x-1} + \sqrt{x+6} &= 7 \\
\sqrt{x+6} &= 7 - \sqrt{x-1} \\
(\sqrt{x+6})^2 &= (7 - \sqrt{x-1})^2 \\
x + 6 &= 49 - 14\sqrt{x-1} + x - 1
\end{align*}
\]

Isolate the remaining radical, divide both sides of the resulting equation by 14, then square both sides of the resulting equation.

\[
\begin{align*}
14\sqrt{x-1} &= 42 \\
\sqrt{x-1} &= 3 \\
(\sqrt{x-1})^2 &= (3)^2 \\
x - 1 &= 9 \\
x &= 10
\end{align*}
\]

The solution \( x = 10 \) agrees with the graphical solution above, but we still need to check.

\[
\begin{align*}
\sqrt{x-1} + \sqrt{x+6} &= 7 \\
\sqrt{10-1} + \sqrt{10+6} &= 7 \\
3 + 4 &= 7
\end{align*}
\]

Thus, \( x = 10 \) is a solution.
27. The calculator’s \textit{intersect} utility provides the following solution.

\[ f(x) = 2 \]
\[ \sqrt{x + 2} + \sqrt{3x + 4} = 2 \]
\[ \sqrt{3x + 4} = 2 - \sqrt{x + 2} \]
\[ (\sqrt{3x + 4})^2 = (2 - \sqrt{x + 2})^2 \]
\[ 3x + 4 = 4 - 4\sqrt{x + 2} + x + 2 \]

Isolate the remaining radical, divide both sides of the resulting equation by 2, then square both sides of the resulting equation.

\[ 4\sqrt{x + 2} = 2 - 2x \]
\[ 2\sqrt{x + 2} = 1 - x \]
\[ (2\sqrt{x + 2})^2 = (1 - x)^2 \]
\[ 4(x + 2) = 1 - 2x + x^2 \]
\[ 4x + 8 = 1 - 2x + x^2 \]

This last equation is nonlinear, so make one side zero, then factor.

\[ 0 = x^2 - 6x - 7 \]
\[ 0 = (x + 1)(x - 7) \]

The solution \( x = -1 \) agrees with the graphical solution above, but we still need to check.

\[ \sqrt{x + 2} + \sqrt{3x + 4} = 2 \]
\[ \sqrt{-1 + 2} + \sqrt{3(-1) + 4} = 2 \cdot 1 + 1 = 2 \]

Thus, \( x = -1 \) is a solution. On the other hand, checking \( x = 7 \),

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\[
\sqrt{x+2} + \sqrt{3x+4} = 2
\]
\[
\sqrt{7+2} + \sqrt{3(7)+4} = 2.
\]

Thus, \(x = 7\) is not a solution.

29.

\[
\sqrt{x+46} = 1 + \sqrt{x-35}
\]
\[
\Rightarrow x + 46 = (1 + \sqrt{x-35})^2
\]
\[
\Rightarrow x + 46 = 1 + 2\sqrt{x-35} + (x - 35)
\]
\[
\Rightarrow 80 = 2\sqrt{x-35}
\]
\[
\Rightarrow 40 = \sqrt{x-35}
\]
\[
\Rightarrow 40^2 = (\sqrt{x-35})^2
\]
\[
\Rightarrow 1600 = x - 35
\]
\[
\Rightarrow x = 1635
\]

31.

\[
\sqrt{x-19} = 13 - \sqrt{x-6}
\]
\[
\Rightarrow x - 19 = (13 - \sqrt{x-6})^2
\]
\[
\Rightarrow x - 19 = 169 - 26\sqrt{x-6} + (x - 6)
\]
\[
\Rightarrow 26\sqrt{x-6} = 182
\]
\[
\Rightarrow \sqrt{x-6} = 7
\]
\[
\Rightarrow (\sqrt{x-6})^2 = 7^2
\]
\[
\Rightarrow x - 6 = 49
\]
\[
\Rightarrow x = 55
\]
33. 
\[ \sqrt{x-2} = 1 + \sqrt{x-49} \]
\[ \Rightarrow x - 2 = (1 + \sqrt{x-49})^2 \]
\[ \Rightarrow x - 2 = 1 + 2\sqrt{x-49} + (x - 49) \]
\[ \Rightarrow 46 = 2\sqrt{x-49} \]
\[ \Rightarrow 23 = \sqrt{x-49} \]
\[ \Rightarrow 23^2 = (\sqrt{x-49})^2 \]
\[ \Rightarrow 529 = x - 49 \]
\[ \Rightarrow x = 578 \]

35. 
\[ \sqrt{x+27} = 1 + \sqrt{x-22} \]
\[ \Rightarrow x + 27 = (1 + \sqrt{x-22})^2 \]
\[ \Rightarrow x + 27 = 1 + 2\sqrt{x-22} + (x - 22) \]
\[ \Rightarrow 48 = 2\sqrt{x-22} \]
\[ \Rightarrow 24 = \sqrt{x-22} \]
\[ \Rightarrow 24^2 = (\sqrt{x-22})^2 \]
\[ \Rightarrow 576 = x - 22 \]
\[ \Rightarrow x = 598 \]

37. 
\[ \sqrt{x+30} = 2 + \sqrt{x-38} \]
\[ \Rightarrow x + 30 = (2 + \sqrt{x-38})^2 \]
\[ \Rightarrow x + 30 = 4 + 4\sqrt{x-38} + (x - 38) \]
\[ \Rightarrow 64 = 4\sqrt{x-38} \]
\[ \Rightarrow 16 = \sqrt{x-38} \]
\[ \Rightarrow 16^2 = (\sqrt{x-38})^2 \]
\[ \Rightarrow 256 = x - 38 \]
\[ \Rightarrow x = 294 \]
39.

\[
\sqrt{x-17} = 10 - \sqrt{x+3} \\
\implies x - 17 = (10 - \sqrt{x+3})^2 \\
\implies x - 17 = 100 - 20\sqrt{x+3} + (x+3) \\
\implies 20\sqrt{x+3} = 120 \\
\implies \sqrt{x+3} = 6 \\
\implies (\sqrt{x+3})^2 = 6^2 \\
\implies x + 3 = 36 \\
\implies x = 33
\]
9.6 Exercises

In Exercises 1-8, state whether or not the given triple is a Pythagorean Triple. Give a reason for your answer.

1. (8, 15, 17)
2. (7, 24, 25)
3. (8, 9, 17)
4. (4, 9, 13)
5. (12, 35, 37)
6. (12, 17, 29)
7. (11, 17, 28)
8. (11, 60, 61)

In Exercises 9-16, set up an equation to model the problem constraints and solve. Use your answer to find the missing side of the given right triangle. Include a sketch with your solution and check your result.

9.

10.

11.

12.

13.
14. 
\[ \text{\small \ diagram of a triangle with sides 12, 4\sqrt{3}} \]

15. 
\[ \text{\small \ diagram of a triangle with sides 5, 10} \]

16. 
\[ \text{\small \ diagram of a triangle with sides 8\sqrt{2}, 8} \]

In Exercises 17-20, set up an equation that models the problem constraints. Solve the equation and use the result to answer the question. Look back and check your result.

17. The legs of a right triangle are consecutive positive integers. The hypotenuse has length 5. What are the lengths of the legs?

18. The legs of a right triangle are consecutive even integers. The hypotenuse has length 10. What are the lengths of the legs?

19. One leg of a right triangle is 1 centimeter less than twice the length of the first leg. If the length of the hypotenuse is 17 centimeters, find the lengths of the legs.

20. One leg of a right triangle is 3 feet longer than 3 times the length of the first leg. The length of the hypotenuse is 25 feet. Find the lengths of the legs.

21. Pythagoras is credited with the following formulae that can be used to generate Pythagorean Triples.

\[
\begin{align*}
a &= m \\
b &= \frac{m^2 - 1}{2}, \\
c &= \frac{m^2 + 1}{2}
\end{align*}
\]

Use the technique of Example 6 to demonstrate that the formulae given above will generate Pythagorean Triples, provided that \( m \) is an odd positive integer larger than one. Secondly, generate at least 3 instances of Pythagorean Triples with Pythagoras’s formula.

22. Plato (380 BC) is credited with the following formulae that can be used to generate Pythagorean Triples.

\[
\begin{align*}
a &= 2m \\
b &= m^2 - 1, \\
c &= m^2 + 1
\end{align*}
\]

Use the technique of Example 6 to demonstrate that the formulae given above will generate Pythagorean Triples, provided that \( m \) is a positive integer larger than 1. Secondly, generate at least 3 instances of Pythagorean Triples with Plato’s formula.
In **Exercises 23-28**, set up an equation that models the problem constraints. Solve the equation and use the result to answer the question. Look back and check your result.

23. Fritz and Greta are planting a 12-foot by 18-foot rectangular garden, and are laying it out using string. They would like to know the length of a diagonal to make sure that right angles are formed. Find the length of a diagonal. Approximate your answer to within 0.1 feet.

24. Angelina and Markos are planting a 20-foot by 28-foot rectangular garden, and are laying it out using string. They would like to know the length of a diagonal to make sure that right angles are formed. Find the length of a diagonal. Approximate your answer to within 0.1 feet.

25. The base of a 36-foot long guy wire is located 16 feet from the base of the telephone pole that it is anchoring. How high up the pole does the guy wire reach? Approximate your answer to within 0.1 feet.

26. The base of a 35-foot long guy wire is located 10 feet from the base of the telephone pole that it is anchoring. How high up the pole does the guy wire reach? Approximate your answer to within 0.1 feet.

27. A stereo receiver is in a corner of a 13-foot by 16-foot rectangular room. Speaker wire will run under a rug, diagonally, to a speaker in the far corner. If 3 feet of slack is required on each end, how long a piece of wire should be purchased? Approximate your answer to within 0.1 feet.

28. A stereo receiver is in a corner of a 10-foot by 15-foot rectangular room. Speaker wire will run under a rug, diagonally, to a speaker in the far corner. If 4 feet of slack is required on each end, how long a piece of wire should be purchased? Approximate your answer to within 0.1 feet.

In **Exercises 29-38**, use the distance formula to find the exact distance between the given points.

29. \((-8, -9)\) and \((6, -6)\)

30. \((1, 0)\) and \((-9, -2)\)

31. \((-9, 1)\) and \((-8, 7)\)

32. \((0, 9)\) and \((3, 1)\)

33. \((6, -5)\) and \((-9, -2)\)

34. \((-9, 6)\) and \((1, 4)\)

35. \((-7, 7)\) and \((-3, 6)\)

36. \((-7, -6)\) and \((-2, -4)\)

37. \((4, -3)\) and \((-9, 6)\)

38. \((-7, -1)\) and \((4, -5)\)

In **Exercises 39-42**, set up an equation that models the problem constraints. Solve the equation and use the result to answer the question. Look back and check your result.

39. Find \(k\) so that the point \((4, k)\) is \(2\sqrt{2}\) units away from the point \((2, 1)\).

40. Find \(k\) so that the point \((k, 1)\) is \(2\sqrt{2}\) units away from the point \((0, -1)\).
41. Find $k$ so that the point $(k,1)$ is $\sqrt{17}$ units away from the point $(2, -3)$.

42. Find $k$ so that the point $(-1, k)$ is $\sqrt{13}$ units away from the point $(-4, -3)$.

43. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Plot the points $P(0, 5)$ and $Q(4, -3)$ on your coordinate system.

a) Plot several points that are equidistant from the points $P$ and $Q$ on your coordinate system. What graph do you get if you plot all points that are equidistant from the points $P$ and $Q$? Determine the equation of the graph by examining the resulting image on your coordinate system.

b) Use the distance formula to find the equation of the graph of all points that are equidistant from the points $P$ and $Q$. Hint: Let $(x, y)$ represent an arbitrary point on the graph of all points equidistant from points $P$ and $Q$. Calculate the distances from the point $(x, y)$ to the points $P$ and $Q$ separately, then set them equal and simplify the resulting equation. Note that this analytical approach should provide an equation that matches that found by the graphical approach in part (a).

44. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Plot the point $P(0, 2)$ and label it with its coordinates. Draw the line $y = -2$ and label it with its equation.

a) Plot several points that are equidistant from the point $P$ and the line $y = -2$ on your coordinate system. What graph do you get if you plot all points that are equidistant from the points $P$ and the line $y = -2$.

b) Use the distance formula to find the equation of the graph of all points that are equidistant from the points $P$ and the line $y = -2$. Hint: Let $(x, y)$ represent an arbitrary point on the graph of all points equidistant from points $P$ and the line $y = -2$. Calculate the distances from the point $(x, y)$ to the points $P$ and the line $y = -2$ separately, then set them equal and simplify the resulting equation.
45. Copy the following figure onto a sheet of graph paper. Cut the pieces of the first figure out with a pair of scissors, then rearrange them to form the second figure. Explain how this proves the Pythagorean Theorem.

46. Compare this image to the one that follows and explain how this proves the Pythagorean Theorem.
1. We check to see if \((8, 15, 17)\) satisfies the Pythagorean Theorem.

\[
8^2 + 15^2 = 17^2 \\
64 + 225 = 289 \\
289 = 289
\]

Thus, \((8, 15, 17)\) is a Pythagorean triple.

3. We check to see if \((8, 9, 17)\) satisfies the Pythagorean Theorem.

\[
8^2 + 9^2 = 17^2 \\
64 + 81 = 281 \\
145 = 289
\]

Thus, \((8, 9, 17)\) is not a Pythagorean triple.

5. We check to see if \((12, 35, 37)\) satisfies the Pythagorean Theorem.

\[
12^2 + 35^2 = 37^2 \\
144 + 1225 = 1369 \\
1369 = 1369
\]

Thus, \((12, 35, 37)\) is a Pythagorean triple.

7. We check to see if \((11, 17, 28)\) satisfies the Pythagorean Theorem.

\[
11^2 + 17^2 = 28^2 \\
121 + 289 = 784 \\
410 = 784
\]

Thus, \((11, 17, 28)\) is not a Pythagorean triple.

9. Let \(x\) represent the missing side of the triangle.

```
x

2

2√3
```

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Use the Pythagorean Theorem to solve for $x$.

$$x^2 = 2^2 + (2\sqrt{3})^2$$

$$x^2 = 4 + 12$$

$$x^2 = 16$$

$$x = 4$$

11. Let $x$ represent the missing side of the triangle.

Use the Pythagorean Theorem to solve for $x$.

$$8^2 = 4^2 + x^2$$

$$64 = 16 + x^2$$

$$48 = x^2$$

$$x = \sqrt{48}$$

$$x = \sqrt{16\sqrt{3}}$$

$$x = 4\sqrt{3}$$

13. Let $x$ represent the missing side of the triangle.

Use the Pythagorean Theorem to solve for $x$.

$$(2\sqrt{3})^2 = x^2 + 2^2$$

$$12 = x^2 + 4$$

$$8 = x^2$$

$$x = \sqrt{8}$$

$$x = \sqrt{4\sqrt{2}}$$

$$x = 2\sqrt{2}$$
15. Let $x$ represent the missing side of the triangle.

Use the Pythagorean Theorem to solve for $x$.

\[
10^2 = 5^2 + x^2 \\
100 = 25 + x^2 \\
75 = x^2 \\
x = \sqrt{75} \\
x = \sqrt{25}\sqrt{3} \\
x = 5\sqrt{3}
\]

17. Let $x$ represent the length of one leg. If the legs are consecutive positive integers, then the second leg would have length $x + 1$.

Use the Pythagorean Theorem to write

\[
5^2 = x^2 + (x + 1)^2.
\]

Expand, make one side zero, then factor.

\[
25 = x^2 + x^2 + 2x + 1 \\
0 = 2x^2 + 2x - 24 \\
0 = x^2 + x - 12 \\
0 = (x + 4)(x - 3)
\]

The solution $x = -4$ must be discarded (the length of a leg must be a positive integer). The second solution, $x = 3$, dictates that the second leg has length $x + 1 = 4$. Hence, the dimensions are $(3, 4, 5)$, which is a well-known Pythagorean Triple.
19. Let $x$ represent the length of one leg. The second leg is one less than twice the first leg, so the second leg has length $2x - 1$.

Use the Pythagorean Theorem to write

$$17^2 = x^2 + (2x - 1)^2.$$ 

Expand, then make one side zero.

$$289 = x^2 + 4x^2 - 4x + 1$$
$$0 = 5x^2 - 4x - 288$$

It is not readily apparent how to factor $(ac = (5)(-288)$ is quite large), so we will use the quadratic formula.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(5)(-288)}}{2(5)} = \frac{4 \pm \sqrt{5776}}{10} = \frac{4 \pm 76}{10}$$

One solution is $x = -72/10 = -7.2$, which we will discard because it is negative. The second solution is $x = 80/10 = 8$, which seems reasonable. If the length of the first leg is $x = 8$, then the length of the second leg is $2x - 1 = 2(8) - 1 = 15$. You can check that $(8, 15, 17)$ forms a Pythagorean Triple.

21. With $m$ and odd integer, let $a = m$, $b = (m^2 - 1)/2$, and $c = (m^2 + 1)/2$. Note that

$$a^2 + b^2 = m^2 + \left(\frac{m^2 - 1}{2}\right)^2$$
$$= m^2 + \frac{m^4 - 2m^2 + 1}{4}$$
$$= \frac{4m^2 + m^4 - 2m^2 + 1}{4}$$
$$= m^4 + 2m^2 + 1.$$ 

On the other hand,

$$c^2 = \left(\frac{m^2 + 1}{2}\right)^2 = \frac{m^4 + 2m^2 + 1}{4}.$$
Hence, $c^2 = a^2 + b^2$, and the expressions for $a$, $b$, and $c$ will produce Pythagorean Triples. For example, the results for $m = 3$, $5$, and $7$ are provided in the following table.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>24</td>
<td>25</td>
</tr>
</tbody>
</table>

23. diagonal $= \sqrt{12^2 + 18^2} = \sqrt{468} \approx 21.63$

25. height $= \sqrt{36^2 - 16^2} = \sqrt{1040} \approx 32.25$

27. length $= 2(3) + \sqrt{13^2 + 16^2} = 6 + \sqrt{425} \approx 26.62$

29. distance $= \sqrt{(6 - (-8))^2 + ((-6) - (-9))^2} = \sqrt{205}$

31. distance $= \sqrt{((-8) - (-9))^2 + (7 - 1)^2} = \sqrt{37}$

33. distance $= \sqrt{((-9) - 6)^2 + ((-2) - (-5))^2} = \sqrt{234} = 3\sqrt{26}$

35. distance $= \sqrt{((-3) - (-7))^2 + (6 - 7)^2} = \sqrt{17}$

37. distance $= \sqrt{((-9) - 4)^2 + (6 - (-3))^2} = \sqrt{250} = 5\sqrt{10}$
39. We want to find \( k \) so that the points \((4, k)\) and \((2, 1)\) are \(2\sqrt{2}\) units apart. The distance \(d\) between \((4, k)\) and \((2, 1)\) is found with the distance formula.

\[
d = \sqrt{(4 - 2)^2 + (k - 1)^2}
\]
\[
d = \sqrt{4 + k^2 - 2k + 1}
\]
\[
d = \sqrt{k^2 - 2k + 5}
\]

Now, set \( d = 2\sqrt{2} \), then square both sides of the resulting equation.

\[
2\sqrt{2} = \sqrt{k^2 - 2k + 5}
\]
\[
(2\sqrt{2})^2 = (\sqrt{k^2 - 2k + 5})^2
\]
\[
8 = k^2 - 2k + 5
\]

This equation is nonlinear, so make one side zero and factor.

\[
0 = k^2 - 2k - 3
\]
\[
0 = (k - 3)(k + 1)
\]

The solution \( k = 3 \) provides \((4, k) = (4, 3)\). The solution \( k = -1 \) provides \((4, k) = (4, -1)\). It is easily checked that both of these points are \(2\sqrt{2}\) units away from the point \((2, 1)\). For example, in the case of the point \((4, 3)\), the distance between \((4, 3)\) and \((2, 1)\) is

\[
d = \sqrt{(4 - 2)^2 + (3 - 1)^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}.
\]

We leave it to readers to check the distance of the second point from \((2, 1)\).

41. We want to find \( k \) so that the points \((k, 1)\) and \((2, -3)\) are \(\sqrt{17}\) units apart. The distance \(d\) between \((k, 1)\) and \((2, -3)\) is found with the distance formula.

\[
d = \sqrt{(k - 2)^2 + (1 - (-3))^2}
\]
\[
d = \sqrt{k^2 - 4k + 4 + 16}
\]
\[
d = \sqrt{k^2 - 4k + 20}
\]

Now, set \( d = \sqrt{17} \), then square both sides of the resulting equation.

\[
\sqrt{17} = \sqrt{k^2 - 4k + 20}
\]
\[
(\sqrt{17})^2 = (\sqrt{k^2 - 4k + 20})^2
\]
\[
17 = k^2 - 4k + 20
\]

This equation is nonlinear, so make one side zero and factor.

\[
0 = k^2 - 4k + 3
\]
\[
0 = (k - 1)(k - 3)
\]

The solution \( k = 1 \) provides \((k, 1) = (1, 1)\). The solution \( k = 3 \) provides \((k, 1) = (3, 1)\). It is easily checked that both of these points are \(\sqrt{17}\) units away from the point \((2, -3)\). For example, in the case of the point \((1, 1)\), the distance between \((1, 1)\) and \((2, -3)\) is
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\[ d = \sqrt{(1 - 2)^2 + (1 - (-3))^2} = \sqrt{1 + 16} = \sqrt{17}. \]

We leave it to readers to check the distance of the second point from \((2, -3)\).

43.

a) In the figure that follows, \(XP = XQ\).

![diagram](image)

It would appear that all points on the line that is the perpendicular bisector of the segment \(PQ\) are equidistant from the points \(P\) and \(Q\). This line appears to go through the origin with slope \(1/2\). Therefore, using \(y = mx + b\), the equation of this line appears to be \(y = (1/2)x + 0\), or more simply, \(y = (1/2)x\).

b) In the image above, we have \(P(0, 5)\), \(Q(4, -3)\), and an arbitrary point \(X(x, y)\) on the perpendicular bisector of the segment joining \(P\) and \(Q\). Thus, the distances \(XP\) and \(XQ\) must be equal. However, the distance \(XP\), the distance between \(X(x, y)\) and \(P(0, 5)\) is given by

\[ XP = \sqrt{(x - 0)^2 + (y - 5)^2} = \sqrt{x^2 + (y - 5)^2}. \]

On the other hand, the distance \(XQ\) between \(X(x, y)\) and \(Q(4, -3)\) is given by

\[ XQ = \sqrt{(x - 4)^2 + (y - (-3))^2} = \sqrt{(x - 4)^2 + (y + 3)^2}. \]

Set these distances equal and square both sides of the resulting equation.

\[ XP = XQ \]
\[ \sqrt{x^2 + (y - 5)^2} = \sqrt{(x - 4)^2 + (y + 3)^2} \]
\[ (\sqrt{x^2 + (y - 5)^2})^2 = (\sqrt{(x - 4)^2 + (y + 3)^2})^2 \]
\[ x^2 + (y - 5)^2 = (x - 4)^2 + (y + 3)^2 \]

Expand.

\[ x^2 + y^2 - 10y + 25 = x^2 - 8x + 16 + y^2 + 6y + 9 \]

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Subtract $x^2$ and $y^2$ from both sides of the equation and simplify.

\[-10y + 25 = -8x + 6y + 25\]

Subtract 25 from both sides of the equation, then solve for $y$.

\[-10y = -8x + 6y\]
\[-10y - 6y = -8x\]
\[-16y = -8x\]

Dividing both sides of this last equation provides $y = (1/2)x$, the same result we got with our graph above.