The Rocket Equation

The Russian/Soviet rocket scientist Konstantin Tsiolkovsky was credited with “discovering” this equation in his 1903 article “Exploration of Cosmic Space by Means of Rocket Devices” in the Russian journal *Science Review*, but various derivations of the equation are known to exist as much as a century earlier. Tsiolkovsky did, in fact, apply the equation to space-going rockets and determined orbital speed.

It seems odd that it would take so long for the equation to come about – after all, it is derived from Newton’s Second Law of Motion, \( F = ma \), which had been around since 1680 with the publication of Principia.

In modern terminology, acceleration is the change in velocity divided by the change in time. Symbolically, the Second Law would then read:

\[
F = m \frac{\Delta v}{\Delta t}
\]

where \( F \) is force, \( m \) is mass, \( \Delta v \) is the change in velocity and \( \Delta t \) is the change in time.

Rearranging the equation, it becomes:

\[
\Delta v = \frac{F}{m} \Delta t
\]

So we can predict the change in velocity of a rocket, given the force the rocket can exert, the mass of the rocket and the length of time it takes to fire the rocket.

Tsiolkovsky, as others before him, had seen the problem with this approach – the mass of the rocket changes as the fuel is used. Newton’s Second Law assumes that the mass of the rocket is constant, and this cannot be.

With an application of calculus, the change of velocity may be calculated more accurately:

\[
\Delta v = v_e \ln \frac{m_{\text{before}}}{m_{\text{after}}}
\]

where \( v_e \) is the velocity of rocket’s exhaust, \( m_{\text{before}} \) is the mass of the rocket with fuel, and \( m_{\text{after}} \) is the mass of the rocket without fuel. The assumption is that the rocket uses up its entire load of fuel in one burn. “\( \ln \)” is the natural logarithm function.

Because of the natural logarithm function, the change in velocity shows a diminishing return for the amount of fuel burned. To illustrate this point, suppose a rocket weighs 2700 kg before launch and 1000 kg after the fuel is burned. Then, \( m_{\text{before}}/m_{\text{after}} = 2.7 \) and the natural logarithm of that fraction is about 1. Thus, \( \Delta v = v_e \); in other words, the rocket’s change in velocity is equal to its exhaust velocity.
But I am not satisfied with that! I want the rocket change its velocity to twice that of its exhaust velocity; in symbols, \( \Delta v = 2 v_e \). So \( \frac{m_{\text{before}}}{m_{\text{after}}} = 2 \) or \( \frac{m_{\text{before}}}{m_{\text{after}}} = 7.4 \). This last step was accomplished by using the exponential \( (e^x) \) function: \( e^2 = 7.4 \).

Thus, the rocket had to weigh 7400 kg before launch (the rocket still weighs 1000 kg after the fuel burns), which is much more than twice its mass, in order to achieve twice the change in speed.